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ALGEBRAIC INVARIANTS

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EDITORS' PREFACE.

The volume called Higher Mathematics, the third edition of which was published in 1900, contained eleven chapters by eleven authors, each chapter being independent of the others, but all supposing the reader to have at least a mathematical training equivalent to that given in classical and engineering colleges. The publication of that volume was discontinued in 1906, and the chapters have since been issued in separate Monographs, they being generally enlarged by additional articles or appendices which either amplify the former presentation or record recent advances. This plan of publication was arranged in order to meet the demand of teachers and the convenience of classes, and it was also thought that it would prove advantageous to readers in special lines of mathematical literature.

It is the intention of the publishers and editors to add other monographs to the series from time to time, if the demand seems to warrant it. Among the topics which are under consideration are those of elliptic functions, the theory of quantics, the group theory, the calculus of variations, and non-Euclidean geometry; possibly also monographs on branches of astronomy, mechanics, and mathematical physics may be included. It is the hope of the editors that this Series of Monographs may tend to promote mathematical study and research over a wider field than that which the former volume has occupied.

PREFACE

THIS introduction to the classical theory of invariants of algebraic forms is divided into three parts of approximately equal length.

Part I treats of linear transformations both from the standpoint of a change of the two points of reference or the triangle of reference used in the definition of the homogeneous coördinates of points in a line or plane, and also from the standpoint of projective geometry. Examples are given of invariants of forms f of low degrees in two or three variables, and the vanishing of an invariant of f is shown to give a geometrical property of the locus $f=0$, which, on the one hand, is independent of the points of reference or triangle of reference, and, on the other hand, is unchanged by projection. Certain covariants such as Jacobians and Hessians are discussed and their algebraic and geometrical interpretations given; in particular, the use of the Hessian in the solution of a cubic equation and in the discussion of the points of inflexion of a plane cubic curve. In brief, beginning with ample illustrations from plane analytics, the reader is led by easy stages to the standpoint of linear transformations, their invariants and interpretations, employed in analytic projective geometry and modern algebra.

Part II treats of the algebraic properties of invariants and covariants, chiefly of binary forms: homogeneity, weight, annihilators, seminvariant leaders of covariants, law of reciprocity, fundamental systems, properties as functions of the roots, and production by means of differential operators. Any quartic equation is solved by reducing it to a canonical form by means of the Hessian (§ 33). Irrational invariants are illustrated by a carefully selected set of exercises (§ 35).

Part III gives an introduction to the symbolic notation of Aronhold and Clebsch. The notation is first explained at length for a simple case; likewise the fundamental theorem on the types of symbolic factors of a term of a covariant of binary forms is first proved for a simple example by the method later used for the general theorem. In view of these and similar attentions to the needs of those making their first acquaintance with the symbolic notation, the difficulties usually encountered will, it is believed, be largely avoided. This notation must be mastered by those who would go deeply into the theory of invariants and its applications.

Hilbert's theorem on the expression of the forms of a set linearly in terms of a finite number of forms of the set is proved and applied to establish the finiteness of a fundamental set of covariants of a system of binary forms. The theory of transvectants is developed as far as needed in the discussion of apolarity of binary forms and its application to rational curves (§§ 53–57), and in the determination by induction of a fundamental system of covariants of a binary form without the aid of the more technical supplementary concepts employed by Gordan. Finally, there is a discussion of the types of symbolic factors in any term of a concomitant of a system of forms in three or four variables, with remarks on line and plane coördinates.

For further developments reference is made at appropriate places to the texts in English by Salmon, Elliott, and Grace and Young, as well as to Gordan's *Invariantentheorie*. The standard work on the geometrical side of invariants is Clebsch-Lindemann, *Vorlesungen über Geometrie*. Reference may be made to books by W. F. Meyer, *Apolarität und Rationale Curve*, *Bericht über den gegenwartigen Stand der Invariantentheorie*, and *Formentheorie*. Concerning invariant-factors, elementary divisors, and pairs of quadratic or bilinear forms, not treated here, see Muth, *Elementartheiler*, Bromwich, *Quadratic Forms and their Classification by Means of Invariant Factors*, and Bôcher's *Introduction to Higher Algebra*. Lack of space prevents also the discussion of the invariants and covariants arising in the

theory of numbers; but an elementary exposition is available in the author's recent book, *On Invariants and the Theory of Numbers*, published, together with Osgood's lectures on functions of several complex variables, by the American Mathematical Society, as *The Madison Colloquium*.

In addition to numerous illustrative examples, there are fourteen sets of exercises which were carefully selected on the basis of experience with classes in this subject.

The author is indebted to Professor H. S. White for suggesting certain additions to the initial list of topics and for reading the proofs of Part I.

CHICAGO, May, 1914.

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ALGEBRAIC INVARIANTS

PART I

ILLUSTRATIONS, GEOMETRICAL INTERPRETATIONS AND APPLICATIONS OF INVARIANTS AND COVARIANTS.

1. Illustrations from Plane Analytics. If x and y are the coördinates of a point in a plane referred to rectangular axes, while x' and y' are the coördinates of the same point referred to axes obtained by rotating the former axes counter-clockwise through an angle θ, then

$$T: \qquad x = x' \cos\theta - y' \sin\theta, \quad y = x' \sin\theta + y' \cos\theta.$$

Substituting these values into the linear function

$$l = ax + by + c,$$

we get $a'x' + b'y' + c$, where

$$a' = a \cos\theta + b \sin\theta, \quad b' = -a \sin\theta + b \cos\theta.$$

It follows that

$$a'^2 + b'^2 = a^2 + b^2.$$

Accordingly, $a^2 + b^2$ is called an *invariant* of l under every transformation of the type T.

Similarly, under the transformation T let

$$L = Ax + By + C = A'x' + B'y' + C,$$

so that

$$A' = A \cos\theta + B \sin\theta, \quad B' = -A \sin\theta + B \cos\theta.$$

By the multiplication * of determinants, we get

$$\begin{vmatrix} a' & b' \\ A' & B' \end{vmatrix} = \begin{vmatrix} a & b \\ A & B \end{vmatrix} \cdot \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = aB-bA,$$

$$\begin{vmatrix} a' & -b' \\ B' & A' \end{vmatrix} = \begin{vmatrix} a & -b \\ B & A \end{vmatrix} \cdot \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = aA+bB.$$

The expressions at the right are therefore invariants of the pair of linear functions l and L under every transformation of type T. The straight lines represented by $l=0$ and $L=0$ are parallel if and only if $aB-bA=0$; they are perpendicular if and only if $aA+bB=0$. Moreover, the quotient of $aB-bA$ by $aA+bB$ is an invariant having an interpretation; it is the tangent of one of the angles between the two lines.

As in the first example, A^2+B^2 is an invariant of L. Between our four invariants of the pair l and L the following identity holds:

$$(aA+bB)^2+(aB-bA)^2=(a^2+b^2)(A^2+B^2).$$

The equation of any conic is of the form $S=0$, where

$$S=ax^2+2bxy+cy^2+2kx+2ly+m.$$

Under the transformation T, S becomes a function of x' and y', in which the part of the second degree

$$F=a'x'^2+2b'x'y'+c'y'^2$$

is derived solely from the part of S of the second degree:

$$f=ax^2+2bxy+cy^2.$$

The coefficient a' of x'^2 is evidently obtained by replacing x by $\cos\theta$ and y by $\sin\theta$ in f, while c' is obtained by replacing x by $-\sin\theta$ and y by $\cos\theta$ in f. It follows at once that

$$a'+c'=a+c.$$

Using also the value of b', we can show that

$$a'c'-b'^2=ac-b^2,$$

* We shall always employ the rule which holds also for the multiplication of matrices: the element in the rth row and sth column of the product is found by multiplying the elements of the rth row of the first determinant by the corresponding elements of the sth column of the second determinant, and adding the products.

but a more general fact will be obtained in § 4 without tedious multiplications. Thus $a+c$ and $d=ac-b^2$ are invariants of f, and also of S, under every transformation of type T. When $S=0$ represents a real conic, not a pair of straight lines, the conic is an ellipse if $d>0$, an hyperbola if $d<0$, and a parabola if $d=0$. When homogeneous coördinates are used, the classifications of conics is wholly different (§ 13).

If x and y are the coördinates of a point referred to rectangular axes and if x' and y' are the coördinates of the same point referred to new axes through the new origin (r, s) and parallel to the former axes, respectively, then

$$t: \qquad x=x'+r, \quad y=y'+s.$$

All of our former expressions which were invariant under the transformations T are also invariant under the new transformations t, since each letter $a, b, \ldots$ involved is invariant under t. But not all of our expressions are invariant under a larger set of transformations to be defined later.

We shall now give an entirely different interpretation to the transformations T and t. Instead of considering (x, y) and (x', y') to be the same point referred to different pairs of coördinate axes, we now regard them as different points referred to the same axes. In the case of t, this is accomplished by translating the new axes, and each point referred to them, in the direction from (r, s) to $(0, 0)$ until those axes coincide with the initial axes. Thus any point (x, y) is translated to a new point (x', y'), where

$$x'=x-r, \quad y'=y-s,$$

both points being now referred to the same axes. Thus each point is translated through a distance $\sqrt{r^2+s^2}$ and in a direction parallel to the directed line from $(0, 0)$ to $(-r, -s)$.

In the case of T, we rotate the new axes about the origin clockwise through angle θ so that they now coincide with the initial axes. Then any point (x, y) is moved to a new point (x', y') by a clockwise rotation about the origin through angle θ. By solving the equations of T, we get

$$x'=x\cos\theta+y\sin\theta, \quad y'=-x\sin\theta+y\cos\theta.$$

These rigid motions (translations, rotations, and combinations of them) preserve angles and distances. But the transformation $x'=2x$, $y'=2y$ is a stretching in all directions from the origin in the ratio $2:1$; while $x'=2x$, $y'=y$ is a stretching perpendicular to the y-axis in each direction in the ratio $2:1$.

From the multiplicity of possible types of transformations, we shall select as the basis of our theory of invariants the very restricted set of transformations which have an interpretation in projective geometry and which suffice for the ordinary needs of algebra.

2. Projective Transformations. All of the points on a straight line are said to form a *range* of points. Project the

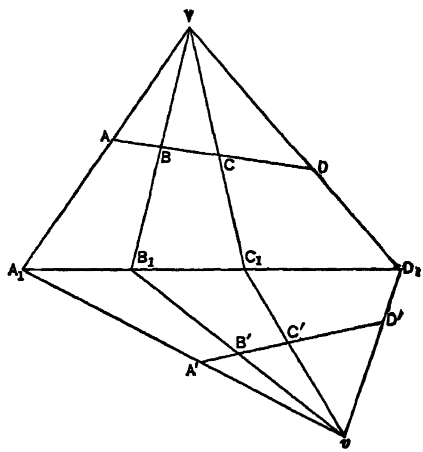

Fig. 1.

points A, B, C, . . . of a range from a point V, not on their line, by means of a pencil of straight lines. This pencil is cut by a new transversal in a range $A_1, B_1, C_1, \ldots$, said to be *perspective* with the range A, B, C, ... Project the points $A_1, B_1, C_1, \ldots$ from a new vertex v by a new pencil and cut it by a new transversal. The resulting range of points A', B', C', . . . is said to be *projective* with the range A, B, C, . . . Likewise, the range obtained by any number of projections and sections is called projective with the given range, and

the one-to-one correspondence thus established between corresponding points of the two ranges is called a *projectivity*.

To obtain an analytic property of a projectivity, we apply the sine proportion to two triangles in Fig. 1 and get

$$\frac{AC}{AV}=\frac{\sin AVC}{\sin ACV}, \quad \frac{BC}{BV}=\frac{\sin BVC}{\sin ACV}.$$

From these and the formulas with D in place of C, we get

$$\frac{AC}{BC}=\frac{AV}{BV}\cdot\frac{\sin AVC}{\sin BVC}, \quad \frac{AD}{BD}=\frac{AV}{BV}\cdot\frac{\sin AVD}{\sin BVD}.$$

Hence, by division

$$\frac{AC}{BC}\div\frac{AD}{BD}=\frac{\sin AVC}{\sin BVC}\div\frac{\sin AVD}{\sin BVD}.$$

The left member is denoted by $(ABCD)$ and is called the *cross-ratio* of the four points taken in this order. Since the right member depends only on the angles at V, it follows that

$$(ABCD)=(A_1B_1C_1D_1),$$

if $A_1, \ldots, D_1$ are the intersections of the four rays by a second transversal. Hence if two ranges are projective, the cross-ratio of any four points of one range equals the cross-ratio of the corresponding points of the other range.

Let each point of the line AB be determined by its distance and direction from a fixed initial point of the line; let a be the resulting coördinate of A, and b, c, x those of B, C, D, respectively. Similarly, let A', B', C', D' have the coördinates a', b', c', x', referred to a fixed initial point on their line. Then

$$(ABCD)=\frac{c-a}{c-b}\div\frac{x-a}{x-b}=\frac{c'-a'}{c'-b'}\div\frac{x'-a'}{x'-b'}=(A'B'C'D').$$

Hence

$$\frac{x'-b'}{x'-a'}=k\frac{x-b}{x-a}, \quad k=\frac{c-a}{c-b}\div\frac{c'-a'}{c'-b'},$$

so that k is a finite constant$\neq 0$, if C is distinct from A and B, and hence C' distinct from A' and B'. Solving for x', we obtain a relation

$$L: \qquad x'=\frac{\alpha x+\beta}{\gamma x+\delta}, \qquad \Delta=\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}\neq 0.$$

In fact,

$$\alpha=b'-ka', \qquad \beta=ka'b-ab', \qquad \gamma=1-k, \qquad \delta=bk-a.$$

If we multiply the elements of the first column of Δ by b and add the products to the elements of the second column, we get

$$\Delta=\begin{vmatrix}b'-ka' & b'(b-a)\\ 1-k & b-a\end{vmatrix}=(b-a)\begin{vmatrix}-ka' & b'\\ -k & 1\end{vmatrix}=k(b-a)(b'-a')\neq 0,$$

if B and A are distinct, so that B' and A' are distinct.

Hence a projectivity between two ranges defines a linear fractional transformation L between the coördinate x of a general point of one range and the coördinate x' of the corresponding point of the other range. The transformation is uniquely determined by the coördinates of three distinct points of one range and those of the corresponding points of the other range. If the ranges are on the same line and if $A'=A$, $B'=B$, $C'=C$, then $k=1$, $\alpha=\delta$, $\beta=\gamma=0$, and $x'=x$. Thus $(ABCD)=(ABCD')$ implies $D'=D$.

Conversely, if L is any given linear fractional transformation (of determinant$\neq 0$) and if each value of x is interpreted as the coördinate of a point on any given straight line l and the value of x' determined by L as the coördinate of a corresponding point on any second given straight line l', the correspondence between the resulting two ranges is a projectivity. This is proved as follows:

Let A, B, C, D be the four points of l whose respective coördinates are four distinct values x_1, x_2, x_3, x_4 of x such that $\gamma x_i+\delta\neq 0$. The corresponding values x_1', x_2', x_3', x_4' of

x' determine four distinct points A', B', C', D' of l'. For, if $i \neq j$,

$$x_i' - x_j' = \frac{\alpha x_i + \beta}{\gamma x_i + \delta} - \frac{\alpha x_j + \beta}{\gamma x_j + \delta} = \frac{\Delta(x_i - x_j)}{(\gamma x_i + \delta)(\gamma x_j + \delta)} \neq 0,$$

$$(A'B'C'D') = \frac{x_3' - x_1'}{x_3' - x_2'} \div \frac{x_4' - x_1'}{x_4' - x_2'} = r\frac{x_3 - x_1}{x_3 - x_2} \div \frac{x_4 - x_1}{x_4 - x_2} = (ABCD)$$

since, if l_i denotes $\gamma x_i + \delta$,

$$r = \left(\frac{\Delta}{l_3 l_1} \Big/ \frac{\Delta}{l_3 l_2}\right) \div \left(\frac{\Delta}{l_4 l_1} \Big/ \frac{\Delta}{l_4 l_2}\right) = 1.$$

If $A' \neq A$, project the points A', B', C', D' from any convenient vertex V' on to any line AB_1 through A and distinct

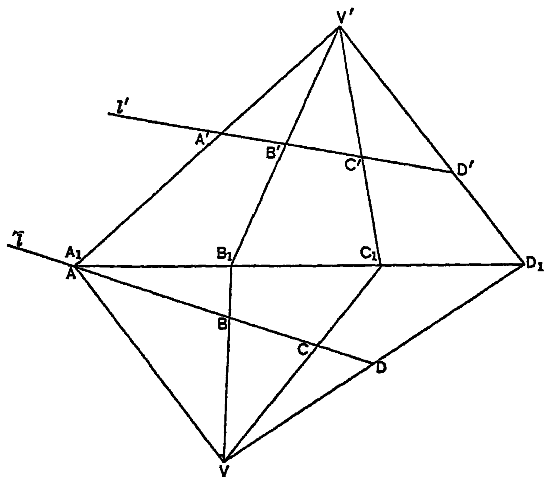

Fig. 2.

from l, obtaining the points $A_1 = A$, B_1, C_1, D_1 of Fig. 2. Let V be the intersection of BB_1 with CC_1 and let VD_1 meet l at P. Then

$$(ABCP) = (A_1B_1C_1D_1) = (A'B'C'D') = (ABCD).$$

From the first and last we have $P = D$, as proved above. Holding x_1, x_2, x_3 fixed, but allowing x_4 to vary, we obtain two projective ranges on l and l'. If $A' = A$, we use l' itself as AB_1 and see that the ranges are perspective.

If l and l' are identical, we first project the range on l' on to a new line ($A'B'$ in Fig. 2) and proceed as before.

Any linear fractional transformation L is therefore a projective transformation of the points of a line or of the points of one line into those of another line. The cross-ratio of any four points is invariant.

3. Homogeneous Coördinates of a Point in a Line. They are introduced partly for the sake of avoiding infinite coördinates. In fact, if $\gamma \neq 0$, the value $-\delta/\gamma$ of x makes x' infinite. We set $x = x_1/x_2$, thereby defining only the ratio of the homogeneous coördinates x_1, x_2 of a point. Let $x' = x_1'/x_2'$. Then, if ρ is a factor of proportionality, L may be given the homogeneous form

$$\rho x_1' = \alpha x_1 + \beta x_2, \qquad \rho x_2' = \gamma x_1 + \delta x_2, \qquad \alpha\delta - \beta\gamma \neq 0.$$

The nature of homogeneous coördinates of points in a line is brought out more clearly by a more general definition. We employ two fixed points A and B of the line as points of reference. We define the homogeneous coördinates of a point P of the line to be any two numbers x, y such that

$$\frac{x}{y} = c\frac{AP}{PB},$$

where c is a constant $\neq 0$, the same for all points P, while AP is a directed segment, so that $AP = -PA$. We agree to take $y = 0$ if $P = B$. Given P, we have the ratio of x to y. Conversely, given the latter ratio, we have the ratio of AP to PB, as well as their sum $AP + PB = AB$, and hence can find AP and therefore locate the point P.

Just as we obtained in plane analytics (*cf.* § 1) the relations between the coördinates of the same point referred to two pairs of axes, so here we desire the values of x and y expressed in terms of the coördinates ξ and η of the same point P referred to new fixed points of reference A', B'. By definition, there is a certain new constant $k \neq 0$ such that

$$\frac{\xi}{\eta} = k\frac{A'P}{PB'}.$$

Since $A'P+PB'=A'B'$, we may replace $A'P$ by $A'B'-PB'$ and get

$$PB'=\frac{k\eta\cdot A'B'}{\xi+k\eta}.$$

Let A have the coördinates ξ', η', referred to A', B'. Then

$$PA=PB'-AB'=PB'-\frac{k\eta'\cdot A'B'}{\xi'+k\eta'}=\frac{(\eta\xi'-\xi\eta')k\cdot A'B'}{(\xi+k\eta)(\xi'+k\eta')}.$$

Similarly, if B has the coördinates ξ_1, η_1, referred to A', B',

$$PB=\frac{(\eta\xi_1-\xi\eta_1)k\cdot A'B'}{(\xi+k\eta)(\xi_1+k\eta_1)}.$$

Hence, by division,

$$\frac{x}{y}=\frac{r(\eta\xi'-\xi\eta')}{s(\eta\xi_1-\xi\eta_1)},\qquad \frac{r}{s}=\frac{-c(\xi_1+k\eta_1)}{\xi'+k\eta'}.$$

Since we are concerned only with the ratio of x to y, we may set

$$x=r\eta'\xi-r\xi'\eta,\qquad y=s\eta_1\xi-s\xi_1\eta.$$

Since the location of A and B with reference to A' and B' is at our choice, as also the constant c (and hence r and s), the values of $r\eta'$ and $-r\xi'$ are at our choice, likewise $s\eta_1$ and $-s\xi_1$. There is, however, the restriction $A\neq B$, whence $\eta'\xi_1\neq\eta_1\xi'$ Thus a change of reference points and constant multiplier c gives rise to a linear transformation

$$T:\qquad x=\alpha\xi+\beta\eta,\qquad y=\gamma\xi+\delta\eta,\qquad \Delta=\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}\neq 0,$$

of coördinates, and conversely every such transformation can be interpreted as the formulas for a change of reference points and constant multiplier.

4. Examples of Invariants. The linear functions

$$l=ax+by,\qquad L=Ax+By$$

become, under the preceding linear transformation T,

$$a(\alpha\xi+\beta\eta)+b(\gamma\xi+\delta\eta)=a'\xi+b'\eta,\qquad A'\xi+B'\eta,$$

where

$$a'=a\alpha+b\gamma,\qquad b'=a\beta+b\delta,\qquad A'=A\alpha+B\gamma,\qquad B'=A\beta+B\delta.$$

Hence the resultant of the new linear functions is

$$\begin{vmatrix} a' & b' \\ A' & B' \end{vmatrix} = \begin{vmatrix} a & b \\ A & B \end{vmatrix} \cdot \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \Delta \begin{vmatrix} a & b \\ A & B \end{vmatrix},$$

and equals the product of the resultant $r=aB-bA$ of the given functions by Δ. Since this is true for every linear homogeneous transformation of determinant Δ, we call r an *invariant* of l and L of *index unity*, the factor which multiplies r being here the first power of Δ.

Employing homogeneous coördinates for points on a line, we see that l vanishes at the single point $(b,-a)$ and that $L=0$ only at $(B,-A)$. These two points are identical if and only if $b:a=B:A$, i.e., if $r=0$. The vanishing of the invariant r thus indicates a geometrical property which is independent of the choice of the points of reference used in defining coördinates on the line; moreover, the property is not changed by a projection of this line from an outside point and a section by a new line. Thus $r=0$ gives a projective property.

Among the present transformations T are the very special transformations given at the beginning of § 1. Of the four functions there called invariants of l and L under those special transformations, r alone is invariant under all of the present transformations. Henceforth the term invariant will be used only when the property of invariance holds for all linear homogeneous transformations of the variables considered.

Our next example deals with the function

$$f=ax^2+2bxy+cy^2.$$

The transformation T (end of § 3) replaces f by

$$F=A\xi^2+2B\xi\eta+C\eta^2,$$

in which

$$A=a\alpha^2+2b\alpha\gamma+c\gamma^2,$$
$$B=a\alpha\beta+b(\alpha\delta+\beta\gamma)+c\gamma\delta,$$
$$C=a\beta^2+2b\beta\delta+c\delta^2.$$

If the discriminant $d=ac-b^2$ of f is zero, f is the square of a linear function of x and y, so that the transformed function

F is the square of a linear function of ξ and η, whence the discriminant $D=AC-B^2$ of F is zero. In other words, $d=0$ implies $D=0$. By inspection, the coefficient of $-b^2$, the highest power of b, in the expansion of D is

$$(\alpha\delta+\beta\gamma)^2-4\alpha\gamma\beta\delta=(\alpha\delta-\beta\gamma)^2=\Delta^2.$$

Thus $D-\Delta^2 d$ is a linear function $bq+r$ of b, where q and r are functions of a, c, α, β, γ, δ. Let a and c remain arbitrary, but give to b the values $\sqrt{ac}$ and $-\sqrt{ac}$ in turn. Since $d=0$ and $D=0$, we have

$$0=\sqrt{ac}\,q+r, \qquad 0=-\sqrt{ac}\,q+r,$$

whence $r=q=0$, $D=\Delta^2 d$. Thus d is an invariant of f of index 2. Another proof is as follows:

$$\Delta^2 d=\begin{vmatrix}\alpha & \gamma\\ \beta & \delta\end{vmatrix}\cdot\begin{vmatrix}a & b\\ b & c\end{vmatrix}\quad\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}$$

$$=\begin{vmatrix}\alpha & \gamma\\ \beta & \delta\end{vmatrix}\cdot\begin{vmatrix}a\alpha+b\gamma & a\beta+b\delta\\ b\alpha+c\gamma & b\beta+c\delta\end{vmatrix}=\begin{vmatrix}A & B\\ B & C\end{vmatrix}=D.$$

We just noted that $d=0$ expresses an algebraic property of f, that of being a perfect square. To give the related geometrical property, employ homogeneous coördinates for the points in a line. Then $f=0$ represents two points which coincide if and only if $d=0$. Thus the vanishing of the invariant d of f expresses a projective property of the points represented by $f=0$.

5. Examples of Covariants. The *Hessian* (named after Otto Hesse) of a function $f(x, y)$ of two variables is defined to be

$$h=\begin{vmatrix}\dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\partial y}\\[2ex] \dfrac{\partial^2 f}{\partial y\partial x} & \dfrac{\partial^2 f}{\partial y^2}\end{vmatrix}.$$

Let f become $F(\xi, \eta)$ under the transformation

$$T:\qquad x=\alpha\xi+\beta\eta, \qquad y=\gamma\xi+\delta\eta, \qquad \Delta=\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}\neq 0.$$

Multiplying determinants according to the rule in § 1, we have

$$h\Delta=\begin{vmatrix} \alpha\dfrac{\partial^2 f}{\partial x^2}+\gamma\dfrac{\partial^2 f}{\partial x\partial y}, & \beta\dfrac{\partial^2 f}{\partial x^2}+\delta\dfrac{\partial^2 f}{\partial x\partial y} \\ \alpha\dfrac{\partial^2 f}{\partial x\partial y}+\gamma\dfrac{\partial^2 f}{\partial y^2}, & \beta\dfrac{\partial^2 f}{\partial x\partial y}+\delta\dfrac{\partial^2 f}{\partial y^2} \end{vmatrix}=\begin{vmatrix} \dfrac{\partial v}{\partial x} & \dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial w}{\partial y} \end{vmatrix},$$

where, by T,

$$(1)\quad v=\alpha\frac{\partial f}{\partial x}+\gamma\frac{\partial f}{\partial y}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}=\frac{\partial F}{\partial \xi},\quad w=\beta\frac{\partial f}{\partial x}+\delta\frac{\partial f}{\partial y}=\frac{\partial F}{\partial \eta}.$$

By the same rule of multiplication of determinants,

$$\begin{vmatrix} \alpha & \gamma \\ \beta & \delta \end{vmatrix}\cdot h\Delta=\begin{vmatrix} \left(\alpha\dfrac{\partial}{\partial x}+\gamma\dfrac{\partial}{\partial y}\right)\dfrac{\partial F}{\partial \xi}, & \left(\alpha\dfrac{\partial}{\partial x}+\gamma\dfrac{\partial}{\partial y}\right)\dfrac{\partial F}{\partial \eta} \\ \left(\beta\dfrac{\partial}{\partial x}+\delta\dfrac{\partial}{\partial y}\right)\dfrac{\partial F}{\partial \xi}, & \left(\beta\dfrac{\partial}{\partial x}+\delta\dfrac{\partial}{\partial y}\right)\dfrac{\partial F}{\partial \eta} \end{vmatrix}.$$

Applying (1) with f replaced by $\partial F/\partial\xi$ for the first column and by $\partial F/\partial\eta$ for the second column, we get

$$\Delta^2 h=\begin{vmatrix} \dfrac{\partial^2 F}{\partial \xi^2} & \dfrac{\partial^2 F}{\partial \xi\partial \eta} \\ \dfrac{\partial^2 F}{\partial \eta\partial \xi} & \dfrac{\partial^2 F}{\partial \eta^2} \end{vmatrix}.$$

Hence the Hessian of the transformed function F equals the product of the Hessian h of the given function f by the square of the determinant of the linear transformation. Consequently, h is called a *covariant of index* 2 of f.

For an interpretation of $h\equiv 0$, see Exs. 4, 5, § 7. In case f is the quadratic function f of § 4, h reduces to $4d$, where d is the invariant $ac-b^2$.

The *functional determinant* or *Jacobian* (named after C. G. J. Jacobi) of two functions $f(x, y)$ and $g(x, y)$ is defined to be

$$\frac{\partial(f, g)}{\partial(x, y)}=\begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{vmatrix}.$$

Let the above transformation T replace f by $F(\xi, \eta)$, and g by $G(\xi, \eta)$. By means of (1), we get

$$\frac{\partial(F, G)}{\partial(\xi, \eta)} = \begin{vmatrix} \alpha\frac{\partial f}{\partial x}+\gamma\frac{\partial f}{\partial y} & \beta\frac{\partial f}{\partial x}+\delta\frac{\partial f}{\partial y} \\ \alpha\frac{\partial g}{\partial x}+\gamma\frac{\partial g}{\partial y} & \beta\frac{\partial g}{\partial x}+\delta\frac{\partial g}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \Delta\frac{\partial(f, g)}{\partial(x, y)}.$$

Hence the Jacobian of f and g is a covariant of index unity of f and g. For example, the Jacobian of the linear functions l and L in § 4 is their resultant r; they are proportional if and only if the invariant r is zero. The last fact is an illustration of the

THEOREM. *Two functions f and g of x and y are dependent if and only if their Jacobian is identically zero.*

First, if $g=\phi(f)$, the Jacobian of f and g is

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \phi'(f)\frac{\partial f}{\partial x} & \phi'(f)\frac{\partial f}{\partial y} \end{vmatrix} \equiv 0.$$

Next, to prove the second or converse part of the theorem, let the Jacobian of f and g be identically zero. If g is a constant, it is a (constant) function of f. In the contrary case, the partial derivatives of g are not both identically zero. Let, for example, $\partial g/\partial x$ be not zero identically. Consider g and y as new variables in place of x and y. Thus $f=F(g, y)$ and the Jacobian is

$$\begin{vmatrix} \frac{\partial F}{\partial g}\frac{\partial g}{\partial x} & \frac{\partial F}{\partial g}\frac{\partial g}{\partial y}+\frac{\partial F}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\partial F}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}.$$

Hence $\partial F/\partial y$ is identically zero, so that F does not involve y explicitly and is a function of g only.

6. Forms and their Classification. A function like ax^3+bx^2y, every term of which is of the same total degree in x and y, is called *homogeneous* in x and y.

A homogeneous rational integral function of $x, y, \ldots$ is called a *form* (or quantic) in $x, y, \ldots$. According as the number of variables is 1, 2, 3, . . . , or q, the form is called *unary*, *binary*, *ternary*, . . . , or *q-ary*, respectively. According as the form is of the first, second, third, fourth, . . . , or pth *order* in the variables, it is called *linear*, *quadratic*, *cubic*, *quartic*, . . . , or *p-ic*, respectively.

For the present we shall deal with binary forms. It is found to be advantageous to prefix binomial coefficients to the literal coefficients of the form, as in the binary quadratic and quartic forms

$$ax^2+2bxy+cy^2, \qquad a_0x^4+4a_1x^3y+6a_2x^2y^2+4a_3xy^3+a_4y^4.$$

7. Definition of Invariants and Covariants of Binary Forms. Let the general binary form f of order p,

$$a_0x^p+pa_1x^{p-1}y+\frac{p(p-1)}{1\cdot 2}a_2x^{p-2}y^2+\ldots+a_py^p,$$

be replaced by

$$A_0\xi^p+pA_1\xi^{p-1}\eta+\frac{p(p-1)}{1\cdot 2}A_2\xi^{p-2}\eta^2+\ldots+A_p\eta^p$$

by the transformation T (§ 5) of determinant $\Delta\neq 0$. If, for every such transformation, a polynomial $I(a_0, \ldots, a_p)$ has the property that

$$I(A_0, \ldots, A_p)\equiv\Delta^\lambda I(a_0, \ldots, a_p),$$

identically in $a_0, \ldots, a_p$, after the A's have been replaced by their values in terms of the a's, then $I(a_0, \ldots, a_p)$ is called an *invariant of index* λ of the form f.

If, for every linear transformation T of determinant $\Delta \neq 0$, a polynomial K in the coefficients and variables in f is such that *

$$K(A_0, \ldots, A_p;\ \xi, \eta) \equiv \Delta^\lambda K(a_0, \ldots, a_p;\ x, y),$$

identically in $a_0, \ldots, a_p, \xi, \eta$, after the A's have been replaced by their values in terms of the a's, and after x and y have been replaced by their values in terms of ξ and η from T, then K is called a *covariant of index* λ of f.

The definitions of invariants and covariants of several binary forms are similar.

These definitions are illustrated by the examples in §§ 4, 5. Note that f itself is a covariant of index zero of f; also that invariants are covariants of order zero.

EXERCISES

1. The Jacobian of $f = ax^2 + 2bxy + cy^2$ and $L = rx + sy$ is

$$J = 2(as - br)x + 2(bs - cr)y.$$

If J is identically zero, $f = tL^2$, where t is a constant. How does this illustrate the last result in § 5? Next, let J be not identically zero. Let k and l be the values of x/y for which $f = 0$; m that for which $L = 0$ and n that for which $J = 0$. Prove that the cross-ratio $(k, m, l, n) = -1$. Thus the points represented by $f = 0$ are separated harmonically by those represented by $L = 0$, $J = 0$.

2. If J is the Jacobian of two binary quadratic forms f and g, the points represented by $J = 0$ separate harmonically those represented by $f = 0$ and also those represented by $g = 0$. Thus $J = 0$ represents the pair of double points of the involution defined by the pairs of points represented by $f = 0$ and $g = 0$.

3. If $f(x, y)$ is a binary form of order n, then (Euler)

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$$

Hint: Prove this for $f = ax^k y^{n-k}$ and for $f = f_1 + f_2$.

4. The Hessian of $(ax + by)^n$ is identically zero.

Hint: It is sufficient to prove this for x^n. Why?

* The factor can be shown to be a power of Δ if it is merely assumed to be a function only of the coefficients of the transformation.

5. Conversely, if the Hessian of a binary form $f(x, y)$ of order n is identically zero, f is the nth power of a linear function.

Hints: The Hessian of f is the Jacobian of $\partial f/\partial x$, $\partial f/\partial y$. By the last result in § 5, these derivatives are dependent:

$$b\frac{\partial f}{\partial x}-a\frac{\partial f}{\partial y}\equiv 0,$$

where a and b are constants. Solving this with Euler's relation in Ex. 3, we get

$$(ax+by)\frac{\partial f}{\partial x}=naf,\ (ax+by)\frac{\partial f}{\partial y}=nbf,$$

$$\frac{\partial \log f}{\partial x}=\frac{na}{ax+by},\ \frac{\partial \log f}{\partial y}=\frac{nb}{ax+by}.$$

Integrating,

$$\log f-n\log(ax+by)=\phi(y)=\psi(x).$$

Hence $\phi=\psi=$constant, say $\log c$. Thus $f=c(ax+by)^n$.

8. Invariants of Covariants. The binary cubic form

$$(1)\qquad f(x, y)=ax^3+3bx^2y+3cxy^2+dy^3$$

has as a covariant of index 2 its Hessian $36\,h$:

$$(2)\qquad h=rx^2+2sxy+ty^2,\quad r=ac-b^2,\quad 2s=ad-bc,\quad t=bd-c^2.$$

Under any linear transformation of determinant Δ, let f become

$$(3)\qquad F=A\xi^3+3B\xi^2\eta+3C\xi\eta^2+D\eta^3.$$

Let H denote the Hessian of F. Then the covariance of h gives

$$(4)\qquad H=R\xi^2+2S\xi\eta+T\eta^2=\Delta^2 h,\qquad R=AC-B^2,\ \ldots$$

Hence $\Delta^2 r$, $2\Delta^2 s$, $\Delta^2 t$ are the coefficients of a binary quadratic form which our transformation replaces by one with the coefficients R, $2S$, T. Since the discriminant of a binary quadratic form is an invariant of index 2,

$$RT-S^2=\Delta^2\{\Delta^2 r\cdot\Delta^2 t-(\Delta^2 s)^2\}=\Delta^6(rt-s^2).$$

Hence $rt-s^2$ is an invariant of index 6 of f.

A like method of proof shows that *any invariant of a covariant of a system of forms is an invariant of the forms.*

As an example in the use of the concepts invariants and covariants in demonstrations, we shall prove that the invariant *

$$(5) \qquad -4(rt-s^2)=(ad-bc)^2-4(ac-b^2)(bd-c^2)$$

is zero if and only if $f(x/y, 1)=0$ has a multiple root, i.e., if $f(x, y)$ is divisible by the square of a linear function of x and y. If the latter be the case, we can transform f into a form (3) with the factor ξ^2; then $C=D=0$ and the function (5) written in capitals is zero, so that the invariant (5) itself is zero. Conversely, if (5) is zero, $f=0$ has a multiple root. For, the Hessian (2) is then a perfect square and hence can be transformed into ξ^2, which, by the covariance of h, differs only by a constant factor from the Hessian $R\xi^2$ of the transformed cubic (3). Thus $S=T=0$. If $D=0$, then $C=0$ (by $T=0$) and (3) has the factor ξ^2, as affirmed. If $D\neq 0$,

$$B=\frac{C^2}{D}, \qquad A=\frac{C^3}{D^2}, \qquad F\equiv D\left(\eta+\frac{C}{D}\xi\right)^3.$$

9. Canonical Form of a Binary Cubic; Solution of Cubic Equations. We shall prove that *every binary cubic form whose discriminant is not zero* † *can be transformed into* X^3+Y^3.

For, if the discriminant (5) of the binary cubic (1) is not zero, the Hessian (2) is the product of two linear functions which are linearly independent. Hence the cubic form f can be transformed into a form F whose Hessian (4) reduces to $2S\xi\eta$, and hence has $R=0$, $T=0$, $S\neq 0$. If $C=0$, then $B=0$ (by $R=0$) and $F=A\xi^3+D\eta^3$, $AD\neq 0$ (by $S\neq 0$). Taking

$$\xi=A^{-\frac{1}{3}}X, \qquad \eta=D^{-\frac{1}{3}}Y,$$

we get $F=X^3+Y^3$, as desired. The remaining case $C\neq 0$ is readily excluded; for, then $B\neq 0$ (by $T=0$) and

$$A=\frac{B^2}{C}, \qquad D=\frac{C^2}{B}, \qquad AD=BC, \qquad S=0.$$

* It is often called the discriminant of f. It equals $-a^4P/27$, where P is the product of the squares of the differences of the roots of $f(x/y,1)=0$. Other writers call a^4P the discriminant of f.

† If zero, f has a square factor and hence can be transformed into X^2Y or X^3.

To solve a cubic equation without a multiple root, we have merely to introduce as new variables the factors ξ and η of the Hessian. For, then, the new cubic is $A\xi^3+D\eta^3=0$.

To treat an example, consider $f=x^3+6x^2y+12xy^2+dy^3=0$. The Hessian is $(d-8)(xy+2y^2)$. Hence we take $\xi=x+2y$ and $\eta=y$ as new variables. We get $f=\xi^3+(d-8)\eta^3$. If $d=9$, we have $\xi^3+\eta^3=0$, whence $\xi/\eta=-1$, $-\omega$ or $-\omega^2$, where ω is an imaginary cube root of unity. But $x/y+2=\xi/\eta$. Hence $x/y=-3$, $-\omega-2$, $-\omega^2-2$.

10. Covariants of Covariants. *Any covariant of a system of covariants of a system of forms is a covariant of the forms.*

The proof of this theorem is similar to that used in the following illustrations. We first show that the Jacobian of a binary cubic form f and its Hessian h is a covariant of index 3 of f. We have

$$\frac{\partial(F, H)}{\partial(\xi, \eta)}=\Delta\frac{\partial(f, \Delta^2 h)}{\partial(x, y)}=\Delta^3\frac{\partial(f, h)}{\partial(x, y)}.$$

As the second illustration we consider the forms f, L in Ex. 1, § 7. Their Jacobian is the double of the covariant $K=vx+wy$ of index unity, where

$$v=as-br, \qquad w=bs-cr.$$

Thus K and L are covariants of the system of forms f, L. These two linear covariants have as an invariant their resultant

$$I=\begin{vmatrix} v & w \\ r & s \end{vmatrix}=as^2-2brs+cr^2.$$

Under a linear transformation of determinant Δ, let f become $A\xi^2+\ldots$, and L become $R\xi+S\eta$. By the covariance of K,

$$V\xi+W\eta=\Delta(vx+wy), \quad V=AS-BR, \quad W=BS-CR.$$

Thus our transformation replaces the linear form having the coefficients Δv and Δw by one having the coefficients V and W. Th. resultant

$$E=\begin{vmatrix} \Delta v & \Delta w \\ r & s \end{vmatrix}$$

of this linear form and L is an invariant of index unity. Hence

$$\begin{vmatrix} V & W \\ R & S \end{vmatrix} = \Delta E, \qquad \begin{vmatrix} V & W \\ R & S \end{vmatrix} = \Delta^2 \begin{vmatrix} v & w \\ r & s \end{vmatrix},$$

so that $I = vs - wr$ is an invariant of index 2 of f and L.

From the earlier expression for I, we see that it is the resultant of f and L. We have therefore illustrated also the theorem that the resultant of any two binary forms is an invariant of those forms.

11. Intermediate Invariants and Covariants. From the invariant $ac - b^2$ of the binary quadratic form

$$f = ax^2 + 2bxy + cy^2$$

we may derive an invariant of the system of forms f and f', where

$$f' = a'x^2 + 2b'xy + c'y^2.$$

Let any linear transformation replace f and f' by

$$F = A\xi^2 + 2B\xi\eta + C\eta^2, \quad F' = A'\xi^2 + 2B'\xi\eta + C'\eta^2.$$

If t is any constant, the form $f + tf'$ is transformed into $F + tF'$. By the invariance of the discriminant of $f + tf'$,

$$(A + tA')(C + tC') - (B + tB')^2 \equiv \Delta^2\{(a + ta')(c + tc') - (b + tb')^2\},$$

identically in t. The equality of the terms free of t states only the known fact that $ac - b^2$ is an invariant of f. Similarly the equality of the terms involving t^2 states merely that $a'c' - b'^2$ is an invariant of f'. But from the terms multiplied by t, we see that

$$(1) \qquad ac' + a'c - 2bb'$$

is an invariant of index 2 of the system of forms f, f'. It is said to be the invariant intermediate between their discriminants. It was discovered by Boole in 1841.

The method is a general one. Let K be any covariant of a form $f(x, y, \ldots)$. Let $a, b, \ldots$ be the coefficients of f. Let $f'(x, y, \ldots)$ be a form of the same order with the coefficients $a', b', \ldots$. If in K we replace a by $a + ta'$, b by $b + tb'$, $\ldots$, and expand in powers of t, we obtain as the

coefficient of any power t^r of t a covariant of the system f, f'. By Taylor's theorem, this covariant is

$$\frac{1}{r!}\left(a'\frac{\partial}{\partial a}+b'\frac{\partial}{\partial b}+\dots\right)^r K, \tag{2}$$

in which the symbolic rth power of $\partial/\partial a$ is to be replaced by $\partial^r/\partial a^r$, etc.

EXERCISES

1. For $r=1$, $K=ac-b^2$, (2) becomes (1).
2. Taking as K the Hessian (2) of cubic (1) in § 8, obtain the covariant

$$(ac'+a'c-2bb')x^2+(ad'+a'd-bc'-b'c)xy+(bd'+b'd-2cc')y^2$$

of index 2 of a pair of binary cubic forms.

3. If (1) is zero, the pair of points given by $f=0$ is harmonic with the pair given by $f'=0$.

12. Homogeneous Coördinates of Points in a Plane. Let

$$L_i: \qquad a_ix+b_iy+c_i=0 \qquad (i=1, 2, 3)$$

be any three linear equations in x, y, such that

$$\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

Interpret x and y as the Cartesian coördinates of a point referred to rectangular axes. Then the equations represent three straight lines L_i forming a triangle. Choose the sign before the radical in

$$p_i=\frac{a_ix+b_iy+c_i}{\pm\sqrt{a_i^2+b_i^2}}$$

so that p_i is positive for a point (x, y) inside the triangle and hence is the length of the perpendicular from that point to L_i. The homogeneous (or trilinear) coördinates of a point (x, y) are three numbers x_1, x_2, x_3 such that

$$\rho x_1=k_1p_1, \qquad \rho x_2=k_2p_2, \qquad \rho x_3=k_3p_3,$$

where k_1, k_2, k_3 are constants, the same for all points. In view of the undetermined common factor ρ, only the ratios of x_1, x_2, x_3 are defined.

For example, let the triangle be an equilateral one with sides of length 2, base on the x-axis and vertex on the y-axis. The equations of the sides L_1, L_2, L_3 are, respectively,

$$\frac{y}{\sqrt{3}}+x=1, \quad \frac{y}{\sqrt{3}}-x=1, \quad y=0.$$

Take each $k_i=1$. Then

$$\rho x_1=\frac{y+\sqrt{3}\,(x-1)}{-2}, \quad \rho x_2=\frac{y-\sqrt{3}\,(x+1)}{-2}, \quad \rho x_3=y.$$

The curve $x_1x_2=x_3^2$ is evidently tangent to L_1(i.e., $x_1=0$) at $Q=(010)$, and tangent to L_2 at $P=(100)$. Substituting for the x_i their values, we see that the Cartesian equation of the curve is

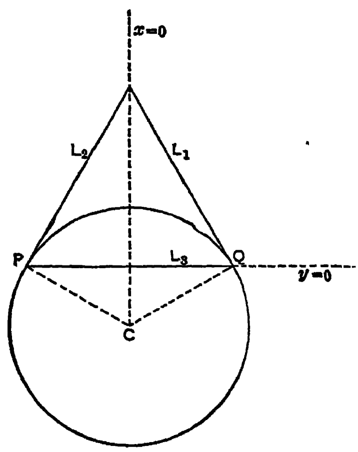

Fig. 3.

$$\tfrac{1}{4}\{(y-\sqrt{3})^2-3x^2\}=y^2 \text{ or } x^2+\left(y+\frac{1}{\sqrt{3}}\right)^2=\frac{4}{3}.$$

Hence it is a circle with radius CP and center at the intersection C of the normal to L_2 at P with the normal to L_1 at Q.

Changing the notation for the coefficients of k_ip_i, call them a_i, b_i, c_i. Then we have

$$(H) \qquad \rho x_i=a_ix+b_iy+c_i, \qquad \Delta\neq 0 \qquad (i=1, 2, 3).$$

Multiply the ith equation by the cofactor A_i of a_i in the determinant Δ and sum for $i=1, 2, 3$. Next use as multiplier the cofactor B_i of b_i; finally, the cofactor C_i of c_i. We get

$$\Delta x=\rho\Sigma A_i x_i, \qquad \Delta y=\rho\Sigma B_i x_i, \qquad \Delta=\rho\Sigma C_i x_i.$$

Hence x and y are rational functions of x_1, x_2, x_3:

$$(C) \qquad x=\frac{A_1x_1+A_2x_2+A_3x_3}{C_1x_1+C_2x_2+C_3x_3}, \qquad y=\frac{B_1x_1+B_2x_2+B_3x_3}{C_1x_1+C_2x_2+C_3x_3}.$$

Any equation $f(x, y)=0$ in Cartesian coördinates becomes, by use of (C), a homogeneous equation $\phi(x_1, x_2, x_3)=0$ in homogeneous coördinates. The reverse process is effected by use of (H). In particular, since any straight line is represented by an equation of the first degree in x and y, it is also represented by a homogeneous equation of the first degree in x_1, x_2, x_3. For example, the sides of the triangle of reference are $x_1=0, x_2=0, x_3=0$. Conversely, any homogeneous equation of the first degree in x_1, x_2, x_3 represents a straight line.

The degree of ϕ is always that of f.

Take the y-axis as L_1, the x-axis as L_2, and let L_3 recede to infinity by making a_3 and b_3 approach zero. Then (H) and (C) become

$$\rho x_1=x, \quad \rho x_2=y, \quad \rho x_3=1; \quad x=\frac{x_1}{x_3}, \quad y=\frac{x_2}{x_3}.$$

We are thus led to a very special, but much used, method of passing from homogeneous to Cartesian coördinates and conversely.

For a new triangle of reference, let the homogeneous coördinates of (x, y) be y_1, y_2, y_3. Then, as in (H),

$$\rho y_i=a'_i x+b'_i y+c'_i \qquad (i=1, 2, 3).$$

Inserting the values of x and y from (C), we get relations like

$$t: \qquad \tau y_i=e_i x_1+f_i x_2+g_i x_3 \qquad (i=1, 2, 3).$$

Hence a change of triangle of reference and constants k_1, k_2, k_3 gives rise to a linear homogeneous transformation t of coördinates. The determinant of the coefficients in t is not

zero, since $y_1=0$, $y_2=0$, $y_3=0$ represent the sides of the new triangle. Conversely, any such transformation t may be interpreted as a change of triangle of reference and constants k_i.

Instead of regarding t as a set of relations between the coördinates of the same point referred to two triangles of reference, we may regard it as defining a correspondence between the points (x_1, x_2, x_3) and (y_1, y_2, y_3) of two different planes, each referred to any chosen triangle of reference in its plane. This correspondence is projective; for, it can be effected by a series of projections and sections, each projection being that of the points of a plane from a point outside of the plane and each section being the cutting of such a bundle of projecting lines by a new plane. Proof will not be given here, nor is the theorem assumed in what follows. It is stated here to show that if I is any invariant of a ternary form f under all linear transformations t, then $I=0$ gives a projective property of the curve $f=0$. It is true conversely that any projective transformation between two planes can be effected by a linear homogeneous transformation on the homogeneous coördinates. Thus for three variables, just as for two (§§ 2, 3), the investigation of the invariants of a form under all linear homogeneous transformations is of especial importance.

13. Properties of the Hessian. Let $f(x_1, \ldots, x_n)$ be a form in the independent variables $x_1, \ldots, x_n$. The Hessian h of f is a determinant of order n in which the elements of the ith row are

$$\frac{\partial^2 f}{\partial x_i \partial x_1}, \quad \frac{\partial^2 f}{\partial x_i \partial x_2}, \quad \ldots, \quad \frac{\partial^2 f}{\partial x_i \partial x_n}.$$

Let f become $\phi(y_1, \ldots, y_n)$ under the transformation

$$T: \qquad x_i = c_{i1}y_1 + c_{i2}y_2 + \ldots + c_{in}y_n \qquad (i=1, \ldots, n),$$

of determinant $\Delta = |c_{ij}|$. The product $h\Delta$ is a determinant of order n in which the element in the ith row and jth column is the sum of the products of the above elements of the ith

row of h by the corresponding elements of the jth column of Δ, and hence is

$$\frac{\partial^2 f}{\partial x_i \partial x_1} c_{1j} + \frac{\partial^2 f}{\partial x_i \partial x_2} c_{2j} + \ldots + \frac{\partial^2 f}{\partial x_i \partial x_n} c_{nj}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_j} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial y_j} + \ldots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_j}\right) = \frac{\partial}{\partial x_i}\frac{\partial \phi}{\partial y_j}.$$

Let Δ' be the determinant obtained from Δ by interchanging its rows and columns. In the product $\Delta' \cdot h\Delta$, the element in the rth row and jth column is therefore

$$c_{1r}\frac{\partial}{\partial x_1}\frac{\partial \phi}{\partial y_j} + \ldots + c_{nr}\frac{\partial}{\partial x_n}\frac{\partial \phi}{\partial y_j} = \frac{\partial}{\partial y_r}\frac{\partial \phi}{\partial y_j},$$

since c_{ir} is the partial derivative of x_i with respect to y_r. Hence

$$\Delta^2 h = \left|\frac{\partial^2 \phi}{\partial y_r \partial y_j}\right|_{r, j=1, \ldots, n} = \text{Hessian of } \phi.$$

Thus h is a covariant of index 2 of f.

To make an application to conics, let f be a ternary quadratic form. Then h is an invariant called the discriminant of f. Let (a_1, a_2, a_3) be a point on $f=0$ (for example, one with $x_3=0$). For $c_{i1}=a_i$ and c_{i2}, c_{i3} chosen so that $\Delta \neq 0$, transformation T makes $(x)=(a)$ correspond to $(y)=(100)$. Hence we may assume that (100) is a point on $f=0$, so that the term in x_1^2 is lacking. Consider the terms $x_1 l$ with the factor x_1. If $l \equiv 0$, f involves only x_2 and x_3 and hence is a product of two linear functions, while $h \equiv 0$. In the contrary case, we may introduce l as a new variable in place of x_2. This amounts to setting $l = x_2$,

$$f = x_1 x_2 + a x_2^2 + b x_2 x_3 + c x_3^2.$$

Replacing x_1 by $x_1 - a x_2 - b x_3$, we get $x_1 x_2 - k x_3^2$, whose Hessian is $2k$. Hence $f=0$ represents two (distinct or coincident) straight lines if and only if the Hessian (discriminant) of f is zero.

Moreover, if the discriminant is not zero, then $k \neq 0$ and we may replace $\sqrt{k}\, x_3$ by x_3 and get $x_1 x_2 - x_3^2$. Hence all conics, which do not degenerate into straight lines, are equivalent

under projective transformation. If the triangle of reference is equilateral and the coördinates are proportional to the perpendiculars upon its sides, $x_1x_2-x_3^2=0$ is a circle (§ 12).

On the contrary, if we employ only translations and rotations, as in plane analytics, there are infinitely many non-equivalent conics; we saw in § 1 that there are then two invariants besides the discriminant.

Next, to make an application to plane cubic curves, let $f(x_1, x_2, x_3)$ be a ternary cubic form. A triangle of reference can be chosen so that $P=(001)$ is a point of the curve $f=0$. Then the term in x_3^3 is lacking, so that

$$f=x_3^2f_1+x_3f_2+f_3,$$

where f_i is a homogeneous function of x_1 and x_2 of degree i. We assume that P is not a singular point, so that the partial derivatives of f with respect to x_1, x_2, and x_3 are not all zero at P. Hence f_1 is not identically zero and can be introduced as a new variable in place of x_1. Thus, after a preliminary linear transformation, we have

$$x_3^2x_1+x_3(ax_1^2+bx_1x_2+cx_2^2)+\bar{f}_3.$$

Replace x_3 by $x_3-\frac{1}{2}(ax_1+bx_2)$. We get

$$F=x_3^2x_1+ex_3x_2^2+C,$$

where C is a cubic function of x_1, x_2, whose second partial derivative with respect to x_i and x_j will be denoted by C_{ij}. The Hessian of F is

$$H=\begin{vmatrix} C_{11} & C_{12} & 2x_3 \\ C_{12} & C_{22}+2ex_3 & 2ex_2 \\ 2x_3 & 2ex_2 & 2x_1 \end{vmatrix}.$$

If the transformation which replaced f by F is of determinant Δ, it replaces the Hessian h of f by $H=\Delta^2h$. Thus $H=0$ represents the same curve as $h=0$, but referred to the same new triangle of reference as $F=0$. We may therefore speak of a definite Hessian curve of the given curve $f=0$. In investigating the properties of these curves we may therefore

refer them to the triangle of reference for which their equations are $H=0$, $F=0$.

The coefficient of x_3^3 in H is evidently $-8e$. Thus P is on the Hessian curve if and only if $e=0$. If d is the coefficient of x_2^3 in C, $x_1=0$ meets $F=0$ at the points for which $x_2^2(ex_3+dx_2)=0$ and these points coincide (at P) if and only if $e=0$. In that case, P is called a *point of inflexion* of $F=0$ and $x_1=0$ the *inflexion tangent* at P. *For a cubic curve $f=0$ without a singular point, every point of inflexion is a point of intersection of the curve with its Hessian curve and conversely.*

14. Inflexion Points and Invariants of a Cubic Curve. Eliminating x_3 between $f=0$, $h=0$, we obtain a homogeneous relation in x_1, x_2, which has therefore at least one set of solutions x'_1, x'_2. For the latter values of x_1 and x_2, $f=0$ and $h=0$ are cubic equations in x_3 with at least one common root, x'_3. Hence $f=0$ has at least one inflexion point (x'_1, x'_2, x'_3). After a suitable linear transformation, this point becomes (001). As in § 13, we can transform f into F, in which e is now zero. If $d=0$, then $F=x_1Q$, and the derivatives

$$\frac{\partial F}{\partial x_1}=Q+x_1\frac{\partial Q}{\partial x_1}, \qquad \frac{\partial F}{\partial x_2}=x_1\frac{\partial Q}{\partial x_2}, \qquad \frac{\partial F}{\partial x_3}=x_1\frac{\partial Q}{\partial x_3}$$

all vanish at an intersection of $x_1=0$, $Q=0$. But we assume that there is no singular point on $f=0$ and thus none on $F=0$. Hence $d\neq 0$. Replacing x_2 by $d^{-\frac{1}{3}}x_2$, we have an F with $d=1$. Adding a multiple of x_1 to x_2, we get

$$F=x_3^2x_1+C, \qquad C=x_2^3+3bx_2x_1^2+ax_1^3,$$

$$H=-4x_3^2C_{22}+2x_1\phi, \qquad \phi=\begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix},$$

so that ϕ is the Hessian of C. By § 8,

$$\phi=36(-b^2x_1^2+ax_1x_2+bx_2^2).$$

Eliminating x_3^2 between $F=0$, $H=0$, we get

$$x_1^2\phi+2C_{22}C=12(x_2^4+6bx_2^2x_1^2+4ax_2x_1^3-3b^2x_1^4)=0.$$

If $x_1=0$, then $x_2=0$ and we obtain the known intersection

(001). For the remaining intersections, we may set $x_1=1$ and obtain from each root r of

$$(1) \qquad r^4+6br^2+4ar-3b^2=0$$

two intersections $(1, r, \pm x'_3)$. For, if $x'_3=0$, then $C=0$, so that (1) would have a multiple root, whence $a^2+4b^3=0$. But the three partial derivatives of F would then all vanish at $(2b, -a, 0)$ or $(1,0,0)$, according as $b\neq 0$ or $b=0$. Hence *there are exactly nine distinct points of inflexion.*

For each of the four roots of (1), the three points of inflexion P and $(1, r, \pm x'_3)$ are collinear, being on $x_2=rx_1$. Since we may proceed with any point of inflexion as we did with P, we see that there are $9\cdot 4/3$ or 12 lines each joining three points of inflexion and such that four of the lines pass through any one of the nine points. The six points of inflexion not on a fixed one of these lines therefore lie by threes on two new lines; three such lines form an *inflexion triangle.* Thus there are $\frac{1}{3}12=4$ inflexion triangles.

The fact that there are four inflexion triangles, one for each root r of (1), can also be seen as follows:

$$\tfrac{1}{24}H+rF=(rx_1-x_2)\{x_3^2-rx_2^2-(r^2+3b)x_1x_2-(r^3+6br+3a)x_1^2\}.$$

The last factor equals

$$x_3^2-\frac{1}{r}\{rx_2+\tfrac{1}{2}(r^2+3b)x_1\}^2,$$

and hence is the product of two linear functions.

Corresponding results hold for any cubic curve $f=0$ without singular points. We have shown that f can be reduced to the special form F by a linear transformation of a certain determinant Δ. Follow this by the transformation which multiplies x_3 by Δ and x_1 by Δ^{-2}, and hence has the determinant Δ^{-1}. Thus there is a transformation of determinant unity which replaces f by a form of type F, and hence replaces the Hessian h of f by the Hessian H of F. Hence there are exactly four values of r for which $\psi=h+24rf$ has a linear factor and therefore three linear factors. These r's are the roots of a quartic (1) in which a and b are functions of the coefficients

of f. To see the nature of these functions, let $x_1-\lambda x_2-\mu x_3$ be a factor of ψ. After replacing x_1 by $\lambda x_2+\mu x_3$ in ψ, we obtain a cubic function of x_2 and x_3 whose four coefficients must be zero. Eliminating λ and μ, we obtain two conditions involving r and the coefficients of f rationally and integrally. The greatest common divisor of their left members is the required quartic function of r. Unless the coefficient of r^4 is constant, a root would be infinite for certain f's. *The inflexion triangles of a general cubic curve $f=0$ are given by $h+24rf=0$, where h is the Hessian of f and r is a root of the quartic* (1) *in which a and b are rational integral invariants of f.*

The explicit expressions for these invariants are very long; they are given in Salmon's *Higher Plane Curves*, §§ 221–2, and were first computed by Aronhold. For their short symbolic expressions, see § 65, Ex. 4.

EXERCISES

1. Using the above inflexion triangle $y_1y_2y_3=0$, where

$$rx_1-x_2=y_1,\ \sqrt{r}x_3\pm(rx_2+kx_1)=2y_2,\ 2y_3,$$
$$k=(r^2+3b)/2,\ \ r^2+k=\tfrac{3}{2}(r^2+b)\neq 0,$$

as shown by use of (1), we have the transformation

$$\sqrt{r}x_3=y_2+y_3,\ (r^2+k)x_1=ry_1+D,\ (r^2+k)x_2=-ky_1+rD,$$

where $D=y_2-y_3$. Using (1) to eliminate a, show that

$$\frac{9}{8}(r^2+b)F=\frac{1}{r}(y_2^3-y_3^3)+3y_1y_2y_3-\frac{1}{8}(r^2+9b)y_1^3.$$

Adding the product of the latter by 54 to its Hessian, we get the product of $y_1y_2y_3$ by $3^5(r^2+b)/r^2$. Hence the nine points of inflexion are found by setting y_1, y_2, y_3 equal to zero in turn.

2. By multiplying the y's in Ex. 1 by constants, derive

$$f=\alpha(z_1^3+z_2^3+z_3^3)+6\beta z_1z_2z_3,$$

called the canonical form. Its Hessian is 6^3h, where

$$h=-\alpha\beta^2(z_1^3+z_2^3+z_3^3)+(\alpha^3+2\beta^3)z_1z_2z_3.$$

Thus find the nine inflexion points and show that the four inflexion triangles are

$$z_1z_2z_3=0,\ \Sigma z_1^3-3lz_1z_2z_3=0\quad (l=1,\ \omega,\ \omega^2),$$

where ω is an imaginary cube root of unity. Their left members are constant multiples of $3h+rf$, where $r=3\beta^2, -(l\alpha-\beta)^2$ are the four roots of (1), with

$$b=\beta(\alpha^3-\beta^3),\ 4a=\alpha^6-20\alpha^3\beta^3-8\beta^6.$$

3. The Jacobian of $f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)$ is

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}.$$

Show that it is a covariant of index unity of $f_1, \ldots, f_n$.

4. Hence the resultant of three ternary linear forms is an invariant of index unity.

5. If $f_1, \ldots, f_n$ are dependent functions, the Jacobian is zero.

PART II

THEORY OF INVARIANTS IN NON-SYMBOLIC NOTATION

15. Homogeneity of Invariants. We saw in § 11 that two binary quadratic forms f and f' have the invariants

$$d=ac-b^2, \qquad s=ac'+a'c-2bb'$$

of index 2. Note that s is of the first degree in the coefficients a, b, c of f and also of the first degree in the coefficients of f', and hence is homogeneous in the coefficients of each form separately. The latter is also true of d, but not of the invariant $s+2d$.

When an invariant of two or more forms is not homogeneous in the coefficients of each form separately, it is a sum of invariants each homogeneous in the coefficients of each form separately.

A proof may be made similar to that used in the following case. Grant merely that $s+2d$ is an invariant of index 2 of the binary quadratic forms f and f'. In the transformed forms (§ 11), the coefficients A, B, C of F are linear in a, b, c; the coefficients A', B', C' of F' are linear in a', b', c'. By hypothesis

$$AC'+A'C-2BB'+2(AC-B^2)=\Delta^2(s+2d).$$

The terms $2d\Delta^2$ of degree 2 in a, b, c on the right arise only from the part $2(AC-B^2)$ on the left. Hence d is itself an invariant of index 2; likewise s itself is an invariant.

However, *an invariant of a single form is always homogeneous*. For example, this is the case with the above discriminant d of f. We shall deduce this theorem from a more general one.

Let I be an invariant of r forms $f_1, \ldots, f_r$ of orders $p_1, \ldots, p_r$ in the same q variables $x_1, \ldots, x_q$. Let a particular term t of I be of degree d_1 in the coefficients of f_1, of degree d_2 in the coefficients of f_2, etc. Apply the special transformation

$$x_1 = \alpha\xi_1, \qquad x_2 = \alpha\xi_2, \ldots, \qquad x_q = \alpha\xi_q,$$

of determinant $\Delta = \alpha^q$. Then f_i is transformed into a form whose coefficients are the products of those of f_i by α^{p_i}. Hence in the function I of the transformed coefficients, the term corresponding to t equals the product of t by

$$(\alpha^{p_1})^{d_1} \ldots (\alpha^{p_r})^{d_r} = \alpha^{\Sigma d_i p_i}.$$

This factor therefore equals Δ^λ, if λ is the index of the invariant. Thus

$$\sum_{i=1}^{r} d_i p_i = \lambda q.$$

Hence $\Sigma d_i p_i$ is constant for all the terms of the invariant.

For the above two quadratic forms, $r = p_1 = p_2 = 2$. For invariant d, we have $d_1 = 2$, $d_2 = 0$, $\Sigma d_i p_i = 4 = 2\lambda$. For s, we have $d_1 = d_2 = 1$, $\Sigma d_i p_i = 4$. Again, the discriminant (§ 8) of the binary cubic form is of constant degree 4 and index $\lambda = 6$; we have $\Sigma d_i p_i = 4 \cdot 3 = 2\lambda$.

If, as in the last example, we take $r = 1$, we see that an invariant of index λ of a single q-ary form of order p is of constant degree d, where $dp = \lambda q$, and hence is homogeneous.

16. Weight of an Invariant I of a Binary Form f. Give to I and f the notations in § 7. Let

$$t = c a_0^{e_0} a_1^{e_1} \ldots a_p^{e_p}$$

be any term of I, and call

$$w = e_1 + 2e_2 + 3e_3 + \ldots + pe_p$$

the *weight* of t. Thus w is the sum of the subscripts of the factors a_i each repeated as often as its exponent indicates. We shall prove that *the various terms of an invariant of a binary form are of constant weight*, and hence call the invariant *isobaric*. For example, $a_0x^2 + 2a_1xy + a_2y^2$ has the invariant $a_0a_2 - a_1^2$, each of whose terms is of weight 2.

To prove the theorem, apply to f the transformation

$$x=\xi, \qquad y=\alpha\eta.$$

We obtain a form with the literal coefficients

$$A_0=a_0, \qquad A_1=a_1\alpha, \qquad A_2=a_2\alpha^2, \ldots, A_p=a_p\alpha^p.$$

Hence if I is of index λ,

$$I(a_0, a_1\alpha, \ldots, a_p\alpha^p)\equiv\alpha^\lambda I(a_0, a_1, \ldots, a_p),$$

identically in α and the a's. The term of the left member which corresponds to the above term t of I is evidently

$$c_0a_0^{e_0} \ldots a_p^{e_p}\alpha^w.$$

Hence $w=\lambda$. The weight of an invariant of degree d of a binary p-ic is thus its index and hence (§ 15) equals $\frac{1}{2}dp$.

17. Weight of an Invariant of any System of Forms. Let $f_1, \ldots, f_n$ be forms in the same variables $x_1, \ldots, x_q$. We define the weight of the coefficient of any term of f_i to be the exponent of x_q in that term, and the weight of a product of coefficients to be the sum of the weights of the factors. For $q=2$, this definition is in accord with that in § 16, where the coefficient a_k of $x_1^{p-k}x_2^k$ was taken to be of weight k. Again, in a ternary quadratic form, the coefficients of x_1^2, x_1x_2 and x_2^2 are of weight zero, those of x_1x_3 and x_2x_3 of weight unity, and that of x_3^2 of weight 2.

Under the transformation of determinant α,

$$x_1=\xi_1, \quad \ldots, \quad x_{q-1}=\xi_{q-1}, \quad x_q=\alpha\xi_q,$$

f_i becomes a form in which the coefficient c' corresponding to a coefficient c of weight k in f_i is $c\alpha^k$. If I is an invariant, $I(c')\equiv\alpha^\lambda I(c)$, identically in α. Hence every term of I is of weight λ.

Thus *any invariant of a single form is isobaric; any invariant of a system of two or more forms is isobaric on the whole, but not necessarily isobaric in the coefficients of each form separately.*

The index equals the weight and is therefore an integer $\geqq 0$.

EXERCISES

1. The invariant $a_0a'_2+a_2a'_0-2a_1a'_1$ of

$$a_0x^2+2a_1xy+a_2y^2, \qquad a'_0x^2+2a'_1xy+a'_2y^2$$

is of total weight 2, but is not of constant weight in a_0, a_1, a_2 alone.

2. Verify the theorem for the Jacobian of two binary linear forms.
3. Verify the theorem for the Hessian of a ternary quadratic form.
4. No binary form of odd order p has an invariant of odd degree d.

18. Products of Linear Transformations. The *product* TT' of

$$T: \qquad x=\alpha\xi+\beta\eta, \qquad y=\gamma\xi+\delta\eta, \qquad \Delta=\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}\neq 0,$$

$$T': \qquad \xi=\alpha'X+\beta'Y, \quad \eta=\gamma'X+\delta'Y, \quad \Delta'=\begin{vmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{vmatrix}\neq 0,$$

is defined to be the transformation whose equations are obtained by eliminating ξ and η between the equations of the given transformations. Hence

$$TT': \quad \begin{cases} x=\alpha''X+\beta''Y, \quad y=\gamma''X+\delta''Y,\\ \alpha''=\alpha\alpha'+\beta\gamma', \beta''=\alpha\beta'+\beta\delta', \gamma''=\gamma\alpha'+\delta\gamma', \delta''=\gamma\beta'+\delta\delta'.\end{cases}$$

Its determinant is seen to equal $\Delta\Delta'$ and hence is not zero. By solving the equations which define T, we get

$$\xi=\frac{\delta}{\Delta}x-\frac{\beta}{\Delta}y, \qquad \eta=\frac{-\gamma}{\Delta}x+\frac{\alpha}{\Delta}y.$$

These equations define the transformation T^{-1} *inverse* to T; each of the products TT^{-1} and $T^{-1}T$ is the *identity* transformation $x=X$, $y=Y$.

The product of transformation T_θ, defined in § 1, by $T_{\theta'}$ is seen to equal $T_{\theta+\theta'}$, in accord with the interpretation given there. The inverse of T_θ is

$$T_{-\theta}: \quad \xi=x\cos\theta+y\sin\theta, \quad \eta=-x\sin\theta+y\cos\theta.$$

Consider also any third linear transformation

$$T_1: \qquad X=\alpha_1U+\beta_1V, \qquad Y=\gamma_1U+\delta_1V.$$

To prove that the associative law

$$(TT')T_1=T(T'T_1)$$

holds, note that the first product is found by eliminating first ξ, η and then X, Y between the equations for T, T', T_1, while the second product is obtained by eliminating first X, Y and then ξ, η between the same equations. Thus the final eliminants must be the same in the two cases.

Hence we may write $TT'T_1$ for either product.

19. Generators of All Binary Linear Transformations. *Every binary linear homogeneous transformation is a product of the transformations*

$$
\begin{array}{lll}
T_n: & x=\xi+n\eta, & y=\eta; \\
S_k: & x=\xi, & y=k\eta \quad (k\neq 0); \\
V: & x=-\eta, & y=\xi.
\end{array}
$$

From these we obtain *

$$
\begin{array}{lll}
V^{-1}=V^3: & x=\eta, & y=-\xi; \\
V^{-1}T_{-n}V=T'_n: & x=x', & y=y'+nx'; \\
V^{-1}S_kV \;=S'_k: & x=kx', & y=y' \quad (k\neq 0).
\end{array}
$$

For $\delta\neq 0$, the transformation T in § 18 equals the product

$$S_\delta S'_{\Delta/\delta}T_{\beta\delta/\Delta}T'_{\gamma/\delta}.$$

For $\delta=0$, so that $\beta\gamma\neq 0$, T equals

$$S_\gamma S'_{-\beta}T_{-\alpha/\beta}V.$$

20. Annihilator of an Invariant of a Binary Form. The binary form in § 7 may be written as either of the sums

$$f=\sum_{i=0}^{p}\binom{p}{i}a_ix^{p-i}y^i=\sum_{i=0}^{p}\binom{p}{i}a_{p-i}x^iy^{p-i}.$$

Transformation V, of determinant unity, replaces the second sum by

$$\sum_{i=0}^{p}\binom{p}{i}a_{p-i}(-1)^i\xi^{p-i}\eta^i.$$

Comparing this with the first sum we see that an invariant of f must be unaltered when

(1) $\qquad a_i$ is replaced by $(-1)^ia_{p-i} \quad (i=0, 1, \ldots, p)$.

* The T's are of the nature of translations, and the S's stretchings.

By § 16, a function $I(a_0, \ldots, a_p)$ is invariant with respect to every transformation S_k if and only if it is isobaric.

Finally, the function must be invariant with respect to every T_n; under this transformation let

$$f=\sum_{i=0}^{p}\binom{p}{i} A_i\xi^{p-i}\eta^i.$$

Differentiating partially with respect to n, we get

$$0=\sum_{i=0}^{p}\binom{p}{i}\left\{\frac{\partial A_i}{\partial n}\xi^{p-i}\eta^i-A_i(p-i)\,\xi^{p-i-1}\,\eta^{i+1}\right\},$$

since $\eta=y$ is free of n, while $\xi=x-n\eta$. The total coefficient of $\xi^{p-j}\eta^j$ is

$$\binom{p}{j}\frac{\partial A_j}{\partial n}-\binom{p}{j-1}(p-j+1)A_{j-1}=0,$$

the second term being absent if $j=0$. But

$$\binom{p}{j}=\binom{p}{j-1}\frac{(p-j+1)}{j}.$$

Hence

$$\frac{\partial A_0}{\partial n}=0,\qquad \frac{\partial A_j}{\partial n}=jA_{j-1}\qquad (j=1, \ldots, p),$$

$$(2)\quad \frac{\partial I(A_0, \ldots, A_p)}{\partial n}=A_0\frac{\partial I}{\partial A_1}+2A_1\frac{\partial I}{\partial A_2}+3A_2\frac{\partial I}{\partial A_3}+\ldots+pA_{p-1}\frac{\partial I}{\partial A_p}.$$

Now $I(a_0, \ldots, a_p)$ is invariant with respect to every transformation T_n, of determinant unity, if and only if

$$I(A_0, \ldots, A_p)\equiv I(a_0, \ldots, a_p),$$

identically in n and the a's. This relation evidently implies

$$\frac{\partial I(A_0, \ldots, A_p)}{\partial n}\equiv 0.$$

Conversely, the latter implies that $I(A_0, \ldots, A_p)$ has the same value for all values of n and hence its value is that given by $n=0$, viz., $I(a_0, \ldots, a_p)$. Hence I has the desired property if and only if the right member of (2) is zero identically in n and the a's. But this is the case if and only if

$$\Omega I(a_0, \ldots, a_p)\equiv 0,$$

identically in the a's, where Ω is the differential operator

$$\Omega=a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+3a_2\frac{\partial}{\partial a_3}+\ldots+pa_{p-1}\frac{\partial}{\partial a_p}.$$

In other words, I must satisfy the partial differential equation $\Omega I=0$. In Sylvester's phraseology, I must be *annihilated* by the operator Ω.

From this section and the preceding we have the important

THEOREM. *A rational integral function I of the coefficients of the binary form f is an invariant of f if and only if I is isobaric, is unaltered by the replacement* (1), *and is annihilated by Ω.*

EXAMPLE

An invariant of degree d of the binary quartic (§ 6) is of weight $2d$ (end of § 16). For $d=1$, the only possible term is ka_2; since $0=\Omega(ka_2)=2ka_1$, we have $k=0$. For $d=2$, we have

$$I=ra_0a_4+sa_1a_3+ta_2^2,$$

$$\Omega I=(s+4r)a_0a_3+(4t+3s)a_1a_2\equiv 0,$$

$$s=-4r,\ t=3r,\quad I=r(a_0a_4-4a_1a_3+3a_2^2).$$

EXERCISES

1. Every invariant of degree 3 of the binary quartic is the product of a constant by

$$J=a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3.$$

2. The invariant of lowest degree of the binary cubic

$$a_0x^3+3a_1x^2y+3a_2xy^2+a_3y^3$$

is its discriminant $(a_0a_3-a_1a_2)^2-4(a_0a_2-a_1^2)(a_1a_3-a_2^2)$.

3. An invariant of two or more binary forms

$$a_0x^{p_1}+\ldots,\quad b_0x^{p_2}+\ldots,\quad c_0x^{p_3}+\ldots$$

is annihilated by the operator

$$\Sigma\Omega\equiv a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+\ldots+b_0\frac{\partial}{\partial b_1}+2b_1\frac{\partial}{\partial b_2}+\ldots+c_0\frac{\partial}{\partial c_1}+\ldots.$$

4. Every invariant of

$$a_0x^2+2a_1xy+a_2y^2,\quad b_0x^2+2b_1xy+b_2y^2$$

of the first degree in the a's and first degree in the b's is a multiple of $a_0b_2+a_2b_0-2a_1b_1$.

5. A binary quadratic and quartic have no such lineo-linear invariant.

6. Find the invariant of partial degrees 2, 1 of a binary linear and a quadratic form.

7. Find the invariant of partial degrees 1, 2 of a binary quadratic and a cubic form.

8. The first two properties in the theorem of § 20 imply that I is homogeneous. For, under replacement (1), any term $ca_0^{e_0}\ldots a_p^{e_p}$ of I, of weight $w=e_1+2e_2+\ldots+pe_p$, implies a term $\pm ca_0^{e_p}a_1^{e_{p-1}}\ldots a_p^{e_0}$ of weight $w=e_{p-1}+2e_{p-2}+\ldots+(p-1)e_1+pe_0$. Adding the two expressions for w, show that the degree $d=e_0+e_1+\ldots+e_p$ is the constant $2w/p$.

21. Homogeneity of Covariants. *A covariant which is not homogeneous in the variables is a sum of covariants each homogeneous in the variables.*

For, if $a, b, \ldots$ are the coefficients of the forms, and K is a covariant,

$$K(A, B, \ldots; \xi, \eta, \ldots)=\Delta^\lambda K(a, b, \ldots; x, y, \ldots).$$

When $x, y, \ldots$ are replaced by their linear expressions in $\xi, \eta, \ldots$, the terms of order ω in $x, y, \ldots$ on the right (and only such terms) give rise to terms of order ω in $\xi, \eta, \ldots$ on the left. Hence, if K_1 is the sum of all of the terms of order ω of K,

$$K_1(A, B, \ldots; \xi, \eta, \ldots)=\Delta^\lambda K_1(a, b, \ldots; x, y, \ldots),$$

and K_1 is a covariant. In this way, $K=K_1+K_2+\ldots$.

Henceforth, we shall restrict attention to covariants which are homogeneous in the variables, and hence of constant *order.*

A covariant K of constant order ω of a single form f is homogeneous in the coefficients, and hence of constant degree d.

For, let f have the coefficients $a, b, \ldots$ and order p, and apply the transformation $x=\alpha\xi$, $y=\alpha\eta, \ldots$. The coefficients of the resulting form are $A=\alpha^p a$, $B=\alpha^p b, \ldots$. Thus

$$K(\alpha^p a, \alpha^p b, \ldots; \alpha^{-1}x, \alpha^{-1}y, \ldots)\equiv(\alpha^q)^\lambda K(a, b, \ldots; x, y, \ldots),$$

identically in $\alpha, a, b, \ldots, x, y, \ldots$, since the left member

equals $K(A, B, \ldots; \xi, \eta, \ldots)$. Now K is homogeneous in $x, y, \ldots$, of order ω; thus

$$\alpha^{-\omega}K(\alpha^p a, \alpha^p b, \ldots; x, y, \ldots) \equiv \alpha^{q\lambda}K(a, b, \ldots; x, y, \ldots).$$

Thus if K has a term of degree d in $a, b, \ldots$, then

$$\alpha^{-\omega}\cdot\alpha^{pd} = \alpha^{q\lambda}, \qquad pd-\omega = q\lambda,$$

so that d is the same for all terms of K.

If f is a form of order p in q variables and if K is a covariant of degree d, order ω and index λ, then $pd-\omega=q\lambda$.

22. Weight of a Covariant of a Binary Form. In

$$f = a_0x^p + pa_1x^{p-1}y + \ldots + \binom{p}{i}a_ix^{p-i}y^i + \ldots + a_py^p$$

the weight of a_k is k. We now attribute the weight 1 to x and the weight 0 to y, so that every term of f is of total weight p.

Apply to f the transformation $x=\xi$, $y=\alpha\eta$. The literal coefficients of the resulting form are

$$A_0 = a_0, \qquad A_1 = \alpha a_1, \ \ldots, \ A_p = \alpha^p a_p.$$

If K is a covariant of degree d, order ω, and index λ, then

$$K(A_0, \ldots, A_p; \ \xi, \eta) = \alpha^\lambda K(a_0, \ldots, a_p; \ x, y).$$

Any term on the left is of the form

$$cA_0^{e_0}A_1^{e_1}\ldots A_p^{e_p}\xi^r\eta^{\omega-r} \qquad (e_0+e_1+\ldots+e_p=d).$$

This equals

$$ca_0^{e_0}a_1^{e_1}\ldots a_p^{e_p}x^ry^{\omega-r}\alpha^{W-\omega} \qquad (W=r+e_1+2e_2+\ldots+pe_p).$$

This must equal a term of the right member, so that $W-\omega=\lambda$. But W is the total weight of that term. Hence every term of K is of the same total weight. *A covariant of index λ and order ω of a binary form is isobaric and its weight is $\omega+\lambda$.*

For a form f of order p in q variables, we attribute the weight 1 to $x_1, x_2, \ldots, x_{q-1}$ and the weight 0 to x_q; then (§ 17) every term of f is of total weight p. By a proof similar to the above, a covariant of index λ and order ω of f is isobaric and its weight is $\omega+\lambda$.

Consider a covariant K homogeneous and of total order ω in the variables $x_1, \ldots, x_q$ of two or more forms f_i. As in § 15, K need not be homogeneous in the coefficients of each form separately, but is a sum of covariants homogeneous in the coefficients of each. Let such a K be of degree d_i in the coefficients of f_i, of order p_i. As in § 21, $\Sigma p_i d_i - \omega = q\lambda$. The total weight of K is $\omega + \lambda$.

For example, if $p_1 = p_2 = q = 2$,

$$f_1 = a_0x^2 + 2a_1xy + a_2y^2, \quad f_2 = b_0x^2 + 2b_1xy + b_2y^2.$$

The Jacobian of f_1 and f_2 is $4K$, where

$$K = (a_0b_1 - a_1b_0)x^2 + (a_0b_2 - a_2b_0)xy + (a_1b_2 - a_2b_1)y^2.$$

Here

$$d_1 = d_2 = 1, \ \omega = 2, \ \lambda = 1, \text{ and } K \text{ is of weight } 3.$$

23. Annihilators of Covariants K of a Binary Form. Proceeding as in § 20, we have instead of (2)

$$\frac{\partial}{\partial n}K(A_0, \ldots, A_p;\ \xi, \eta) = \sum_{j=0}^{p} \frac{\partial K}{\partial A_j}\frac{\partial A_j}{\partial n} + \frac{\partial K}{\partial \xi}\frac{\partial \xi}{\partial n} + \frac{\partial K}{\partial \eta}\frac{\partial \eta}{\partial n}$$

$$= \sum_{j=1}^{p} jA_{j-1}\frac{\partial K}{\partial A_j} - \eta\frac{\partial K}{\partial \xi},$$

and obtain the following result: K is covariant with respect to every transformation $x = \xi + n\eta$, $y = \eta$, if and only if it is annihilated by *

(1) $$\Omega - y\frac{\partial}{\partial x} \qquad \left(\Omega = a_0\frac{\partial}{\partial a_1} + \ldots + pa_{p-1}\frac{\partial}{\partial a_p}\right).$$

The binary form is unaltered if we interchange x and y, a_i and a_{p-i} for $i = 0, 1, \ldots, p$. Hence K is covariant with respect to every transformation $x = \xi$, $y = \eta + n\xi$, if and only if it is annihilated by

(2) $$O - x\frac{\partial}{\partial y} \qquad \left(O \equiv pa_1\frac{\partial}{\partial a_0} + (p-1)a_2\frac{\partial}{\partial a_1} + \ldots + a_p\frac{\partial}{\partial a_{p-1}}\right).$$

Denote a covariant of order ω of the binary p-ic by

$$K = Sx^\omega + S_1x^{\omega-1}y + \ldots + S_\omega y^\omega.$$

* For another derivation, see the corollary in § 47.

By operating on K by (2), we must have

$$(OS-S_1)x^\omega+(OS_1-2S_2)x^{\omega-1}y+\ldots+(OS_{\omega-1}-\omega S_\omega)xy^{\omega-1}+OS_\omega y^\omega\equiv 0,$$

identically in x, y. Hence K becomes

$$(3)\quad K=Sx^\omega+OSx^{\omega-1}y+\tfrac{1}{2}O^2Sx^{\omega-2}y^2+\ldots+\frac{1}{\omega!}O^\omega Sy^\omega,$$

while, by $OS_\omega=0$,

$$(4)\qquad O^{\omega+1}S=0.$$

Hence a covariant is uniquely determined by its *leader* S. (Cf. § 25).

Similarly, K is annihilated by (1) if and only if

$$(5)\quad \Omega S=0,\quad \Omega S_1=\omega S,\quad \Omega S_2=(\omega-1)S_1,\quad \ldots,\quad \Omega S_\omega=S_{\omega-1}.$$

The function S of $a_0, \ldots, a_p$ must be homogeneous and isobaric (§§ 21, 22). If such a function S is annihilated by Ω, it is called a *seminvariant*. If we have S_ω, we may find $S_{\omega-1}$ by (5), then $S_{\omega-2}, \ldots$, and finally S_1. But if K is a covariant, we can derive S_ω from S. For, by § 20, the transformation $x=-\eta$, $y=\xi$ replaces f by a form in which $A_i=(-1)^i a_{p-i}$; by the covariance of K,

$$S(A)\xi^\omega+\ldots=S(A)y^\omega+\ldots\equiv S(a)x^\omega+\ldots+S_\omega(a)y^\omega,$$

so that $S_\omega(a)=S(A)$. Hence S_ω is derived from S by the replacement (1) in § 20.

When the seminvariant leader S is given, and hence also ω (see Ex. 1), the function (3) is actually a covariant of f; likewise the function whose coefficients are given by (5). Proof will be made in § 25. In the following exercises, indirect verification of the covariance is indicated.

EXERCISES

1. The weight of the leader S of a covariant of order ω of a binary form f is $W-\omega=\lambda$ and hence (§ 21) is $\frac{1}{2}(pd-\omega)$. Thus S and f determine ω.

2. The binary cubic has the seminvariant $S=a_0a_2-a_1^2$. A covariant with S as leader of is order $\omega=2$ and is

$$(a_0a_2-a_1^2)x^2+(a_0a_3-a_1a_2)xy+(a_1a_3-a_2^2)y^2.$$

Since this is the Hessian of the cubic, it is a covariant.

3. Find the covariant of the binary cubic f whose leader is $a_0^2a_3-3a_0a_1a_2+2a_1^3$, the only seminvariant of weight 3 and degree 3. It is the Jacobian of f and its Hessian.

4. A covariant of two or more binary forms is annihilated by

$$\Sigma\Omega-y\frac{\partial}{\partial x}, \qquad \Sigma O-x\frac{\partial}{\partial y}.$$

5. Find a seminvariant of weight 2 and partial degrees 1, 1 of a binary quadratic and cubic. Show that it is the leader of the covariant

$$(a_0b_2-2a_1b_1+a_2b_0)x+(a_0b_3-2a_1b_2+a_2b_1)y.$$

24. Alternants. Consider the annihilators

$$\Omega=\sum_{j=1}^{p} ja_{j-1}\frac{\partial}{\partial a_j}=\sum_{k=0}^{p-1}(k+1)a_k\frac{\partial}{\partial a_{k+1}},$$

$$O=\sum_{j=1}^{p}(p-j+1)a_j\frac{\partial}{\partial a_{j-1}}=\sum_{k=0}^{p-1}(p-k)a_{k+1}\frac{\partial}{\partial a_k}$$

of invariants of a binary form. We have

$$\Omega O=\sum_{j=1}^{p} ja_{j-1}\left\{(p-j+1)\frac{\partial}{\partial a_{j-1}}+\sum_{k=0}^{p-1}(p-k)a_{k+1}\frac{\partial^2}{\partial a_j\partial a_k}\right\},$$

$$O\Omega=\sum_{k=0}^{p-1}(p-k)a_{k+1}\left\{(k+1)\frac{\partial}{\partial a_{k+1}}+\sum_{j=1}^{p} ja_{j-1}\frac{\partial^2}{\partial a_k\partial a_j}\right\}.$$

The terms involving second derivatives are identical. Hence

$$\Omega O-O\Omega=\sum_{i=0}^{p-1}(i+1)(p-i)a_i\frac{\partial}{\partial a_i}-\sum_{i=1}^{p} i(p-i+1)a_i\frac{\partial}{\partial a_i}$$

$$=\sum_{i=0}^{p}(p-2i)a_i\frac{\partial}{\partial a_i},$$

since the first sum is the first sum in ΩO with j replaced by $i+1$, and the second is the first sum in $O\Omega$ with k replaced by $i-1$.

If S is a homogeneous function of $a_0, \ldots, a_p$ of total degree d and hence a sum of terms

$$ca_0^{e_0}a_1^{e_1}\ldots a_p^{e_p} \qquad (e_0+e_1+\ldots+e_p=d),$$

we readily verify Euler's theorem:

$$\sum_{i=0}^{p}\frac{\partial S}{\partial a_i}=dS.$$

If S is isobaric, it is a sum of terms

$$t = ca_0^{e_0}a_1^{e_1} \dots a_p^{e_p} \qquad (e_1+2e_2+\dots+pe_p=w)$$

where w is constant; then

$$\sum_{i=0}^{p} ia_i\frac{\partial t}{\partial a_i}=\sum_{i=0}^{p} ie_it=wt, \qquad \sum_{i=0}^{p} ia_i\frac{\partial S}{\partial a_i}=wS.$$

Hence *if S is both homogeneous (of degree d) and isobaric (of weight w) in $a_0, \dots, a_p$, then*

(1) $$(\Omega O - O\Omega)S=\omega S, \qquad \omega = pd-2w.$$

A covariant with the leader S has the order ω. (Ex. 1, § 23.)

Since OS is of degree d and weight $w+1$, we have

$$\begin{aligned}(\Omega O^2-O^2\Omega)S &\equiv (\Omega O-O\Omega)OS+O(\Omega O-O\Omega)S\\ &=(\omega-2)OS+\omega OS=2(\omega-1)OS.\end{aligned}$$

Hence for $r=1$ and $r=2$, we have

(2) $$(\Omega O^r-O^r\Omega)S=r(\omega-r+1)O^{r-1}S.$$

To proceed by induction, note that (2) implies

$$\begin{aligned}(\Omega O^{r+1}-O^{r+1}\Omega)S &\equiv (\Omega O^r-O^r\Omega)OS+O^r(\Omega O-O\Omega)S\\ &=r(\omega-2-r+1)O^rS+\omega O^rS=(r+1)(\omega-r)O^rS,\end{aligned}$$

so that (2) holds also when r is replaced by $r+1$.

25. Seminvariants as Leaders of Binary Covariants.

Lemma. *If S is a seminvariant, not identically zero, of degree d and weight w, of a binary p-ic, then $dp-2w \geqq 0$.*

Suppose on the contrary that S is a seminvariant for which $\omega<0$, where $\omega=dp-2w$. By the definition of a seminvariant, $\Omega S=0$. Hence, by (2), § 24,

(1) $$\Omega O^rS=r(\omega-r+1)O^{r-1}S \qquad (r=1, 2, 3, \dots)$$

and no one of the coefficients on the right is zero. But

$$O^{dp-w+1}S\equiv 0,$$

being of degree d and weight $dp+1$; in fact, the largest weight of a function of $a_0, \dots, a_p$ of degree d is dp, the weight of a_p^d. Then (1) for $r=dp-w+1$ gives $O^{dp-w}S=0$. Then (1)

for $r=dp-w$ gives $O^{dp-w-1}S=0$, etc. Finally, we get $S=0$, contrary to hypothesis.

THEOREM. *There exists a covariant K of a binary p-ic whose leader is any given seminvariant S of the p-ic.*

The covariant K is in fact given by (3), § 23. By (1), for $r=\omega+1$,

$$\Omega O^{\omega+1}S=0.$$

Hence $O^{\omega+1}S$ is a seminvariant of degree d and weight

$$w'=w+\omega+1=pd-w+1.$$

Then $dp-2w'=-(pd-2w)-2$ is negative. Hence (4), § 23, follows from the Lemma. Thus K is annihilated by the operator (2), § 23. Next, in

$$\left(\Omega-y\frac{\partial}{\partial x}\right)K,$$

the coefficient of $x^{\omega-r}y^r$ is

$$\frac{1}{r!}\Omega O^r S-\frac{1}{(r-1)!}(\omega-r+1)O^{r-1}S\equiv\frac{1}{r!}\{\Omega O^r S-r(\omega-r+1)O_1^{r-1}S\},$$

which is zero by (1). Hence K is covariant with respect to all of the transformations T_n and T'_n of § 19. Now

$$T_{-1}T'_1T_{-1}=V:\qquad x=-Y,\qquad y=X,$$

as shown by eliminating ξ, η, ξ_1, η_1 between

$$\begin{cases} x=\xi-\eta, \\ y=\quad\ \eta, \end{cases}\qquad \begin{cases} \xi=\quad\ \xi_1, \\ \eta=\eta_1+\xi_1, \end{cases}\qquad \begin{cases} \xi_1=X-Y, \\ \eta_1=\quad\ Y. \end{cases}$$

Since K is of constant weight, it is covariant with respect to every S_k (§ 16). Hence, by § 19, K is covariant with respect to all binary linear transformations.

26. Number of Linearly Independent Seminvariants.

LEMMA. *Given any homogeneous isobaric function S of $a_0, \ldots, a_p$ of degree d and weight w, where $\omega\equiv dp-2w>0$, we can find a homogeneous isobaric function S_1 of degree d and weight $w+1$ such that $\Omega S_1\equiv S$.*

In (2), § 24, replace S by $\Omega^{r-1}S$, whose degree is d and weight is $w-r+1$, so that its ω is $\omega+2r-2$. We get

$$\Omega O^r\Omega^{r-1}S-O^r\Omega^r S=r(\omega+r-1)O^{r-1}\Omega^{r-1}S.$$

Multiply this by

$$(-1)^{r-1}\frac{1}{r!\omega(\omega+1)\ .\ .\ .\ (\omega+r-1)}.$$

The new right member cancels the second term of the new left member after r is replaced by $r-1$ in the latter. Hence if we sum from $r=1$ to $r=w+1$, the terms not cancelling are those from the first terms of the left members, that from the right member for $r=1$, and that from the second term on the left for $r=w+1$. But the last is zero, since $\Omega^{w+1}S\equiv 0$, $\Omega^w S$ being of weight zero and hence a power of a_0. Hence we get $\Omega S_1\equiv S$, where

$$S_1\equiv\sum_{r=1}^{w+1}\frac{(-1)^{r-1}}{r!\omega(\omega+1)\ .\ .\ .\ (\omega+r-1)}O^r\Omega^{r-1}S.$$

THEOREM.* *The number of linearly independent seminvariants of degree d and weight w of the binary p-ic is zero if $pd-2w<0$, but is*

$$(w;\ d,\ p)-(w-1;\ d,\ p),$$

if $pd-2w\geq 0$, where $(w;\ d,\ p)$ denotes the number of partitions of w into d integers chosen from $0,\ 1,\ .\ .\ .\ ,\ p$, with repetitions allowed.

If $p\geq 4$, $(4;\ 2,\ p)=3$, since $4+0$, $3+1$, $2+2$ are the partitions of 4 into 2 integers. Also, $(3;\ 2,\ p)=2$, corresponding to $3+0$, $2+1$. Hence the theorem states that every seminvariant of degree 2 and weight 4 of the binary p-ic, $p\geq 4$, is a numerical multiple of one such (see the Example in § 20).

The literal part of any term of a seminvariant S specified in the theorem is a product of d factors chosen from a_0, a_1, . . . , a_p, with repetitions allowed, such that the sum of the subscripts of the d factors is w. Hence there are $(w;\ d,\ p)$ possible terms. Giving them arbitrary coefficients and operating on the sum of the resulting terms with Ω, we obtain a linear combination S' of the $(w-1;\ d,\ p)$ possible products

* Stated by Cayley; proved much later by Sylvester.

of degree d and weight $w-1$. By the Lemma there exists * an S for which ΩS is any assigned S'. Thus the coefficients of our $S' \equiv \Omega S$ are arbitrary and hence are linearly independent functions of the $(w;\ d,\ p)$ coefficients of S. Hence the condition $\Omega S \equiv 0$ imposes $(w-1;\ d,\ p)$ linearly independent linear relations between the coefficients of S and hence determines $(w-1;\ d,\ p)$ of the coefficients of S in terms of the remaining coefficients. Thus the difference gives the number of arbitrary constants in the general seminvariant S, and hence the number of linearly independent seminvariants S.

27. Hermite's Law of Reciprocity. Consider any partition

$$w = n_1 + n_2 + \ldots + n_\delta$$

of w into $\delta \leqq d$ positive integers such that $p \geqq n_1 \geqq n_2 \quad \ldots \quad \geqq n_\delta$. Write n_1 dots in a row; then in a second row write n_2 dots under the first n_2 dots of the first row; then in a third row write n_3 dots under the first n_3 dots of the second row, etc., until w dots have been written in δ rows.

Now count the dots by columns instead of by rows. The number m_1 of dots in the first (left-hand) column is δ; the number m_2 in the second column is $\leqq m_1$; etc. The number of columns is $n_1 \leqq p$. Hence we have a partition

$$w = m_1 + m_2 + \ldots + m_\pi$$

of w into $\pi \leqq p$ positive integers not exceeding d.

Hence to every one of the $(w;\ d,\ p)$ partitions of the first kind corresponds a unique one of the $(w;\ p,\ d)$ partitions of the second kind. The converse is true, since we may begin with an arrangement in columns and read off an arrangement by rows. The correspondence is thus one-to-one. Hence $(w;\ d,\ p) = (w;\ p,\ d)$.

By two applications of this result, we get

$$(w;\ d,\ p) - (w-1;\ d,\ p) = (w;\ p,\ d) - (w-1;\ p,\ d).$$

Hence, by the theorem of § 26, *the number of linearly independent*

* Provided $pd - 2(w-1) > 0$, which holds if $pd - 2w \geqq 0$. But if $pd - 2w < 0$, our theorem is true by the Lemma in § 25.

seminvariants of weight w and degree d of the binary p-ic equals the number of weight w and degree p of the binary d-ic.

Let $dp-2w\equiv\omega\geqq 0$. Then, by the theorem of § 25, each seminvariant in question uniquely determines a covariant of order ω.

The number of linearly independent covariants of degree d and order ω of the binary p-ic equals the number of linearly independent covariants of degree p and order ω of the binary d-ic.

The covariants are of course invariants if and only if $\omega=0$.

EXERCISES

1. Show by means of (1), § 24, that $w=\frac{1}{2}pd$ for an invariant.

2. Show that $(6;\ 6, 3)=7$, $(5;\ 6, 3)=5$. Find the two linearly independent seminvariants of weight 6 and degree 6 of the binary cubic.

3. There are only two linearly independent seminvariants of degree 4 and weight 4 of a binary quartic. Find them.

4. There is a single invariant or no invariant of degree 3 of the binary p-ic according as p is or is not a multiple of 4. (Cayley.)

Hint: Every invariant of the binary cubic is a product of a constant by a power of its discriminant, of order 4 (§ 30).

5. The binary p-ic has a single covariant or no covariant of order p and degree 2 according as p is or is not a multiple of 4. (Cayley.)

Hint: Every covariant of the binary quadratic f is of the type $c\ D^nf^m$, where c is a constant and D the discriminant of f (§ 29.) The degree $2n+m$ of the product equals its order $2m$ if $m=2n$. Thus f has a covariant of order and degree p if and only if $p=4n$, viz., $c\ D^nf^{2n}$.

6. No covariant of degree 2 has a leader of odd weight.

7. If S is of degree d_1 in the coefficients of a binary p_1-ic, of degree d_2 in the coefficients of a p_2-ic, . . . , and of total weight w, (2), § 24, holds with Ω and O replaced by $\Sigma\Omega$ and ΣO, and ω replaced by Σp_id_i-2w. For any such S, there exists an S_1 of partial degrees d_i and total weight $w+1$ for which $(\Sigma\Omega)S_1=S$. If S is a seminvariant, $\omega\geqq 0$. Generalize §§ 26, 27, using $(w;\ d_1,\ p_1;\ d_2,\ p_2;\ .\ .\ .)$ to denote the number of ways in which w can be expressed as a sum of d_1 or fewer positive integers $\leqq p_1$, of d_2 or fewer positive integers $\leqq p_2$, etc.

FUNDAMENTAL SYSTEM OF COVARIANTS OF A BINARY FORM, §§ 28–31

28. Certain Seminvariants. For $a_0 \neq 0$, we may set

$$f = a_0x^p + pa_1x^{p-1}y + \ldots + a_py^p = a_0(x-\alpha_1y) \ldots (x-\alpha_py).$$

Apply to f the transformation

$$T_n: \qquad x = \xi + n\eta, \qquad y = \eta.$$

Then each root α_i of $f=0$ is diminished by n, since

$$x - \alpha_iy = \xi - (\alpha_i - n)\eta.$$

Hence the difference of any two roots is unaltered.

In particular, if $n = -a_1/a_0$, f is transformed into the reduced form

$$f' = a_0\xi^p + \binom{p}{2}a'_2\xi^{p-2}\eta^2 + \binom{p}{3}a'_3\xi^{p-3}\eta^3 + \ldots,$$

where

$$a'_2 = a_2 - \frac{a_1^2}{a_0}, \qquad a'_3 = a_3 - 3\frac{a_1a_2}{a_0} + 2\frac{a_1^3}{a_0^2}, \ldots,$$

and the roots of $f'=0$ are $\alpha_i + a_1/a_0$ $(i=1, \ldots, p)$ Since

$$\alpha_i + \frac{a_1}{a_0} = \alpha_i - \frac{\Sigma\alpha_1}{p} = \frac{(\alpha_i-\alpha_1) + \ldots + (\alpha_i - \alpha_p)}{p},$$

each root of $f'=0$ is a linear function of the differences of the roots of $f=0$ and hence is unaltered by every transformation T_n. The same is true of a'_2/a_0, a'_3/a_0, . . . , which equal numerical multiples of the elementary symmetric functions of the roots of $f'=0$. Hence the polynomials

$$A_2 = a_0a'_2 = a_0a_2 - a_1^2,$$
$$A_3 = a_0^2a'_3 = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3,$$
$$A_4 = a_0^3a'_4 = a_0^3a_4 - 4a_0^2a_1a_3 + 6a_0a_1^2a_2 - 3a_1^4$$

are homogeneous and isobaric,* and are invariants of f with respect to all transformations T_n. By definition they are, therefore, seminvariants of f provided the subscript of each A in question does not exceed p.

* This is evident for A_2, A_3, A_4. Further A's will not be employed here. A general proof follows from § 34.

Since f' was derived from f by a linear transformation of determinant unity, any seminvariant S of f has the property

$$S(a_0, \ldots, a_p)=S(a_0,0,a'_2, \ldots, a'_p)=S\left(a_0, 0, \frac{A_2}{a_0}, \ldots, \frac{A_p}{a_0^{p-1}}\right).$$

Hence any rational integral seminvariant is the quotient of a polynomial in $a_0, A_2, \ldots, A_p$ by a power of a_0. For $p \leqq 4$, we shall find which of these quotients equal rational integral functions of $a_0, \ldots, a_p$ and hence give rational integral seminvariants. The method is due to Cayley.

For $p=1$, S is evidently a numerical multiple of a power of a_0. Since a_0 is the leader of the covariant $f=a_0x+a_1y$ of f, we conclude that every covariant of a binary linear form f is a product of a power of f by a constant; in particular, there is no invariant.

29. Binary Quadratic Form. Since A_2 does not have the factor a_0, we conclude that every rational integral seminvariant is a polynomial in a_0 and A_2. Now A_2 is an invariant of f (§ 4), and a_0 is the leader of the covariant f of f. Hence *a fundamental system of rational integral covariants of the binary quadratic form f is given by f and its discriminant A_2.* We express in these words our result that any such covariant is a rational integral function of f and A_2.

30. Binary Cubic Form. We seek a polynomial $P(a_0, A_2, A_3)$ with the implicit, but not explicit, factor a_0. Write A'_i for the terms of A_i free of a_0:

$$(1) \qquad A'_2=-a_1^2, \qquad A'_3=2a_1^3.$$

We desire that $P(0, A'_2, A'_3)\equiv 0$, identically in a_1. Now

$$4A'^3_2+A'^2_3\equiv 0,$$

$$(2) \qquad 4A_2^3+A_3^2\equiv a_0^2D,$$

where D is the discriminant of the cubic form,

$$D=a_0^2a_3^2-6a_0a_1a_2a_3+4a_0a_2^3+4a_1^3a_3-3a_1^2a_2^2.$$

By means of (2) we eliminate A_3^2 and higher powers of A_3 from $P(a_0, A_2, A_3)$ and conclude that any seminvariant is of the form π/a_0^k, where π is a polynomial in a_0, A_2, A_3, D, of degree 1 or 0 in A_3. If $k>0$, we may assume that not every term of π has the explicit factor a_0. In the latter case, π does not have the implicit factor a_0. For, if it did,

$$\pi' = \pi(0, A'_2, A'_3, D') \equiv 0, \qquad D' = 4a_1^3a_3 - 3a_1^2a_2^2.$$

Since a_3 occurs in D', but not in A'_2 or A'_3, π' is free of D'. By (1), the first power of A'_3 is not cancelled by a power of A'_2. Hence π' is free of A'_3 and hence of A'_2.

A fundamental system of rational integral seminvariants of the binary cubic is given by a_0, A_2, A_3, D. They are connected by the syzygy (2).

A fundamental system of rational integral covariants of the binary cubic f is given by f, its discriminant D, its Hessian H, and the Jacobian J of f and H. They are connected by the syzygy

$$4H^3 + J^2 \equiv f^2D. \tag{3}$$

The last theorem follows from the first one and (2), since a_0, A_2, A_3 are the leaders of the covariants f, H, J.

31. Binary Quartic Form. We first seek polynomials $P(a_0, A_2, A_3, A_4)$ with the implicit, but not explicit, factor a_0. Thus

$$P' = P(0, A'_2, A'_3, A'_4) \equiv 0, \; A'_2 = -a_1^2, \; A'_3 = 2a_1^3, \; A'_4 = -3a_1^4.$$

The simplest P' is evidently $3A'^2_2 + A'_4$. We get

$$A_4 + 3A_2^2 = a_0^2I, \qquad I = a_0a_4 - 4a_1a_3 + 3a_2^2.$$

We drop A_4 and consider polynomials $\pi(a_0, A_2, A_3, I)$ with the implicit, but not explicit, factor, a_0. Such a polynomial is given by (2), § 30. For $a_0 = 0$, $D = -a_1^2I = A'_2I$. We have

$$A_2I - D = a_0J,$$

$$J = a_0a_2a_4 - a_0a_3^2 + 2a_1a_2a_3 - a_1^2a_4 - a_2^3.$$

Eliminating D between this relation and (2), § 30, we get

$$a_0^3J - a_0^2A_2I + 4A_2^3 + A_3^2 = 0. \tag{1}$$

In view of their origin, I and J are seminvariants of the quartic f. Since they are unaltered by the replacement (1), § 20, they are invariants of f (*cf.* § 20, Example and Ex. 1). In view of (1), π equals a polynomial ϕ in a_0, A_2, A_3, I, J, of degree 0 or 1 in A_3. Suppose that ϕ does not have the explicit factor a_0. Then the equal function of $a_0, \ldots, a_4$ is not divisible by a_0. For, if it were,

$$\phi(0, -a_1^2, 2a_1^3, 3a_2^2-4a_1a_3, -a_1^2a_4+\ldots)\equiv 0.$$

In view of the term a_4, ϕ cannot involve J, and hence not I. Nor can ϕ be linear in A_3 in view of the odd power a_1^3. Hence ϕ is free of A_3 and hence of A_2.

A fundamental system of rational integral seminvariants of the binary quartic is given by a_0, A_2, A_3, I, J. *They are connected by the syzygy* (1).

A fundamental system of rational integral covariants of the binary quartic f *is given by* f, *its invariants* I *and* J, *its Hessian* H *and the Jacobian* G *of* f *and* H. *They are connected by the syzygy*

$$f^3J-f^2HI+4H^3+G^2\equiv 0. \tag{2}$$

The second theorem follows from the first one, since a_0, A_2, A_3 are the leaders of the covariants f, H, G.

It would be excessively laborious, if not futile, to apply the same method to the binary quintic, whose fundamental system is composed of 23 covariants,* most of which are very complex. The symbolic method is here superior both as to theory and as to compact notation (see Part III.).

Canonical Form of Binary Quartic. Solution of Quartic Equations

32. Theorem. *A binary quartic form* f, *whose discriminant is not zero, can be transformed linearly into the canonical form*

$$X^4+Y^4+6mX^2Y^2. \tag{1}$$

* Faà di Bruno, *Theorie der Binären Formen*, German tr. by Walter, 1881, pp. 199, 316–355. Salmon, *Modern Higher Algebra*, Fourth Edition, 1885, p. 227, p. 347.

The reason there is here a parameter m lies in the existence of two invariants I and J of weights (and hence indices) 4 and 6, and hence a rational *absolute* invariant I^3/J^2, i.e., one of index zero, and consequently having the same value for f and any form derived from f by linear transformation.

Since f vanishes for four values of x/y and hence is the product of four linear functions, it can be expressed (in three ways) as a product of two quadratic forms, say those in the right members of the next equations. To prove our theorem it suffices to show that there exist constant p, q, r, s (each $\neq 0$) and α, β ($\alpha \neq \beta$) such that

$$p(x+\alpha y)^2+q(x+\beta y)^2 \equiv ax^2+2bxy+cy^2,$$
$$r(x+\alpha y)^2+s(x+\beta y)^2 \equiv gx^2+2hxy+ky^2.$$

For, the product f of these becomes (1) by the transformation

$$X=\sqrt[4]{pr}\,(x+\alpha y), \qquad Y=\sqrt[4]{qs}\,(x+\beta y),$$

of determinant $\neq 0$. The conditions for the two identities are

$$p+q=a, \qquad p\alpha+q\beta=b, \qquad p\alpha^2+q\beta^2=c,$$
$$r+s=g, \qquad r\alpha+s\beta=h, \qquad r\alpha^2+s\beta^2=k.$$

The first three equations are consistent if

$$\begin{vmatrix} 1 & 1 & a \\ \alpha & \beta & b \\ \alpha^2 & \beta^2 & c \end{vmatrix} \div (\beta-\alpha) \equiv c-b(\alpha+\beta)+a\alpha\beta=0.$$

If $p=0$, or if $q=0$, the same equations give $b^2=ac$, so that the first quadratic factor of f and hence f would have a double root. Similarly, the last three equations have solutions $r\neq 0$, $s\neq 0$, if

$$k-h(\alpha+\beta)+g\alpha\beta=0.$$

If the determinant $ah-bg$ is not zero, the last two relations determine $\alpha+\beta$ and $\alpha\beta$, and hence give α and β as the roots of *

$$(ah-bg)z^2-(ak-cg)z+bk-ch=0.$$

* Its left member is obtained by setting $x/y=-z$ in the Jacobian of the two quadratic factors of f.

If its roots were equal, the two relations would give

$$c-2b\alpha+a\alpha^2=0, \qquad k-2h\alpha+g\alpha^2=0,$$

and the two quadratic factors of f would vanish for $x/y=-\alpha$.

If $ah-bg=0$, but $ch-bk\neq 0$, we interchange x with y and proceed as before. If both determinants vanish, either $b\neq 0$ and the second quadratic factor is the product of the first by h/b, or else $b=0$ and hence $h=0$ and no transformation of f is needed.

33. Actual Determination of the Canonical Quartic. Let Δ denote the determinant of the coefficients of x, y in X, Y. Then f, its invariants I and J and Hessian H are related to the canonical form, its invariants and Hessian, as follows:

$$f=X^4+Y^4+6mX^2Y^2,$$

$$I=\Delta^4(1+3m^2), \qquad J=\Delta^6(m-m^3),$$

$$H=\Delta^2\{m(X^4+Y^4)+(1-3m^2)X^2Y^2\}.$$

Thus $\Delta^2 m$ may be found from the resolvent cubic equation

$$4(\Delta^2 m)^3-I(\Delta^2 m)+J=0.$$

Then Δ^4 may be found from I. We may select either square root as Δ^2 and hence find m. In fact, by replacing X by $X\sqrt{-1}$ in f, the signs of Δ^2 and m are changed. By eliminating X^4+Y^4, we get

$$\Delta^2 mf-H\equiv\Delta^2(9m^2-1)X^2Y^2.$$

If $9m^2=1$, f is the square of $X^2\pm Y^2$ and the discriminant of f would vanish. Hence we obtain XY by a root extraction. Thus X and Y are determined up to constant factors t and t^{-1}. We may find t by comparing the coefficients of x^4 and x^3y in f and the expansion of its canonical form, or by use of the Jacobian G of f and H:

$$G=\Delta^3(1-9m^2)XY(X^4-Y^4),$$

and combining the resulting X^4-Y^4 with the earlier X^4+Y^4. Or from f and XY we can find X^2+Y^2 and then $X\pm Y$.

To solve $f=0$, we have only to find the canonical form

Seminvariants, Invariants, and Covariants of a Binary Form f as Functions of the Roots of $f=0$, §§ 34–37.

34. Seminvariants in Terms of the Roots. Give f the notation used in § 28, so that $\alpha_1, \ldots, \alpha_p$ are the roots of $f=0$. After removing possible factors a_0 from a given seminvariant of f, we obtain a seminvariant S not divisible by a_0. Let δ be the degree of the homogeneous function S of the a's. Thus S is the product of a_0^δ by a polynomial in $a_1/a_0, \ldots, a_p/a_0$ of degree δ. The latter equal numerical multiples of the elementary symmetric functions of $\alpha_1, \ldots, \alpha_p$, each of which is linear in every root. Hence our polynomial equals a symmetric polynomial σ in $\alpha_1, \ldots, \alpha_p$ of degree δ in every root.

Since S is of constant weight w and since a_i/a_0 equals a function of total degree i in the roots, σ is homogeneous in the roots and of total degree w in them.

Besides being homogeneous and isobaric in the a's, a seminvariant must be unaltered by every transformation T_n of § 28. Under that transformation, each root is diminished, by n (§ 28). Since

$$\alpha_i=\alpha_1+(\alpha_i-\alpha_1) \qquad (i=2, \ldots, p)$$

we can express σ as a polynomial $P(\alpha_1)$ whose coefficients are rational integral functions of the differences of the roots. If $P(\alpha_1)$ is of degree $\geqq 1$ in α_1, we have $P(\alpha_1)=P(\alpha_1-n)$, for all values of n. But an equation in n cannot have an infinitude of roots. Hence $P(\alpha_1)$ does not involve α_1, so that σ equals a polynomial in the differences of the roots.

Multiplying by the factors a_0 removed, we obtain the theorem:

Any seminvariant of degree d and weight w of the binary form $a_0x^p+\ldots$ equals the product of a_0^d by a rational integral symmetric function σ of the roots, homogeneous (of total degree w) in the roots, of degree $\leqq d$ in any one root, and expressible as a polynomial in the differences of the roots.

Conversely, *any such product can be expressed as a polynomial in the a's and this polynomial is a seminvariant.*

Since the factor σ is symmetric in the roots, and is of degree $\leqq d$ in any one root, its product by $a_0{}^d$ equals a homogeneous polynomial in the a's whose degree is d. This polynomial is isobaric since σ is homogeneous, and is unaltered by every transformation T_n, since σ is expressible as a function of the differences of the roots.

The importance of these theorems is due mainly to the fact that they enable us to tell by inspection (without computation by annihilators) whether or not a given function of the roots and a_0 is a seminvariant. A like remark applies to the theorem in § 35 on invariants and that in § 36 on covariants.

EXAMPLE

The binary cubic has the seminvariant

$$a_0{}^2\sum_3(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)=a_0{}^2(\Sigma\alpha_1{}^2-\Sigma\alpha_1\alpha_2)$$
$$=a_0{}^2\{(\Sigma\alpha_1)^2-3\Sigma\alpha_1\alpha_2\}=a_0{}^2\left\{\left(\frac{-3a_1}{a_0}\right)^2-3\left(\frac{3a_2}{a_0}\right)\right\}=-9(a_0a_2-a_1{}^2).$$

35. Invariants in Terms of the Roots. A seminvariant of f is an invariant of f if and only if it is unaltered by the transformation $x=-\eta$, $y=\xi$ (§ 20). For the latter,

$$x-\alpha y=-\alpha\left(\xi+\frac{1}{\alpha}\eta\right),$$

so that α_r is replaced by $-1/\alpha_r$, and hence $\alpha_r-\alpha_s$ by

$$\frac{\alpha_r-\alpha_s}{\alpha_r\alpha_s}.$$

The coefficient of ξ^p in the transformed binary form is

$$A_0=(-1)^p\alpha_1\alpha_2\ \ldots\ \alpha_p a_0.$$

By § 34, any seminvariant of f is of the type

$$a_0{}^d\Sigma c_i(\text{product of } w \text{ factors like } \alpha_r-\alpha_s).$$

Hence this is an invariant if and only if it equals

$$(-1)^{pd}(\alpha_1\ \ldots\ \alpha_p)^d a_0{}^d\Sigma c_i\left(\text{product of the } w \text{ corresponding } \frac{\alpha_r-\alpha_s}{\alpha_r\alpha_s}\right),$$

and hence if $\pm\alpha_1^d \ldots \alpha_p^d$ equals the product of the factors $\alpha_r\alpha_s$ in the denominators. This is the case if and only if each root occurs exactly d times in every term of the sum and if pd is even. By the total number of α's, $pd=2w$.

Any invariant of degree d and weight w of the binary form $a_0x^p+\ldots$ equals the product of a_0^d by a sum of products of constants and certain differences of the roots, such that each root occurs exactly d times in every product; moreover, the sum equals a homogeneous symmetric function of the roots of total degree w. Conversely, the product of any such sum by a_0^d equals a rational integral invariant.

EXERCISES

1. $a_0^2(\alpha_1-\alpha_2)^2$ is an invariant of the binary quadratic form. Any invariant is a numerical multiple of a power of this one.

2. $a_0^2\sum_3(\alpha_1-\alpha_2)^2(\alpha_3-\alpha_4)^2$ is an invariant of the binary quartic.

3. $a_0^2\sum_3(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)$ is not an invariant of the binary cubic.

4. If we multiply $a_0^{2(p-1)}$ by the product of the squares of the differences of the roots of the binary p-ic f, we obtain an invariant (discriminant of f). Also verify that $pd=2w$.

5. The sum of the coefficients of any seminvariant is zero.

Hint: Use $f=(x+y)^p$, whose roots are all equal.

6. Every invariant of the binary cubic is a power of its discriminant.

7. A function which satisfies the conditions in the theorem of § 35 except that of symmetry in the roots is called an *irrational invariant.* If $\alpha_1, \ldots, \alpha_4$ are the roots of a binary quartic f, and

$$u=(\alpha_1-\alpha_4)(\alpha_2-\alpha_3),\quad v=(\alpha_2-\alpha_4)(\alpha_3-\alpha_1),\quad w=(\alpha_1-\alpha_2)(\alpha_3-\alpha_4),$$

why are a_0u, a_0v, a_0w irrational invariants of f? They are the roots of $z^3-12Iz-\delta=0$, where δ^2 is the product of a_0^6 by the product of the squares of the differences of the roots and hence is the discriminant of f. Hints: $u+v+w=0$, and $s=uv+uw+vw$ is a symmetric function of $\alpha_1, \ldots, \alpha_4$ in which each α_i occurs twice in every product of differences, so that a_0^2s is an invariant of degree 2. By the Example in § 20, $a_0^2s=cI$, where c is a constant. To determine c, take $\alpha_1=1, \alpha_2=-1, \alpha_3=2, \alpha_4=-2$, so that $f=(x^2-y^2)(x^2-4y^2)$, $I=73/12$, $u=-9$, $v=1$, $w=8$, $s=-73$. Hence $c=-12$. As here, so always an irrational algebraic invariant is a root of an equation whose coefficients are rational invariants.

8. If α_1, α_2 are the roots of the binary quadratic form f, and α_3, α_4 the roots of f' in § 11, the simultaneous invariant

$$ac'+a'c-2bb'=aa'\{\alpha_3\alpha_4+\alpha_1\alpha_2-\tfrac{1}{2}(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)\}=\tfrac{1}{2}a_0(u-v),$$

if the product ff' is identified with the quartic in Ex. 7. Hence a simultaneous invariant of the quadratic factors of a quartic is an irrational invariant of the quartic. Why *a priori* is the invariant three-valued?

9. The cross-ratios of the four roots of the quartic are $-v/u$, etc. These six are equal in sets of three if $I=0$. For, if $s=0$,

$$vw=u(-v-w)=u^2,\quad uw=v(-u-w)=v^2,\quad \frac{-v}{u}=\frac{-u}{w}=\frac{-w}{v}.$$

The remaining three are the reciprocals of these and are equal.

10. By Ex. 3, § 11, one of the cross-ratios is -1 if $ac'+\ldots=0$. Why does this agree with Ex. 8?

11. The product of the squares of the differences of the roots of the cubic equation in Ex. 7 is known * to be

$$-4(-12I)^3-27\delta^2=a_0^6(u-v)^2(u-w)^2(v-w)^2.$$

Also,* $\delta^2=256(I^3-27J^2)$. Hence the left member becomes $3^6\cdot 4^4J^2$. Thus

$$3^3\cdot 4^2J=\pm a_0^3(u-v)(u-w)(v-w).$$

Using J from § 31, and the special values in Ex. 7, show that the sign is plus. Verify that the cross-ratios equal $-1, -1, 2, 2, \frac{1}{2}, \frac{1}{2}$, if $J=0$.

36. Covariants in Terms of the Roots. Let $K(a_0,\ldots,a_p;x,y)$ be a covariant of constant degree d (in the coefficients) and constant order ω (in the variables) of the binary form $f=a_0x^p+\ldots$ Then

$$K=a_0{}^dy^\omega\kappa,$$

where κ is a polynomial in x/y and the roots $\alpha_1,\ldots,\alpha_p$ of $f=0$. Under the transformation T_n in § 28, let f become $A_0\xi^p+\ldots$, with the roots $\alpha'_1,\ldots,\alpha'_p$. Then

$$\frac{x}{y}-\alpha_i=\frac{\xi}{\eta}-\alpha'_i,\qquad \alpha_r-\alpha_s=\alpha'_r-\alpha'_s.$$

Making use of the identities

$$\frac{x}{y}=\left(\frac{x}{y}-\alpha_1\right)+\alpha_1,\qquad \alpha_i=(\alpha_i-\alpha_1)+\alpha_1,$$

* *Cf.* Dickson, *Elementary Theory of Equations*, p. 33, p. 42, Ex. 7.

we see that κ equals a polynomial $P(\alpha_1)$ whose coefficients are rational integral functions of the differences of $x/y, \alpha_1, \ldots, \alpha_p$ in pairs. Since

$$K(A_0, \ldots, A_p;\ \xi, \eta) = K(a_0, \ldots, a_p;\ x, y), \qquad A_0 = a_0, \quad \eta = y,$$

we have $$\kappa\left(\alpha'_1, \ldots, \alpha'_p, \frac{\xi}{\eta}\right) = \kappa\left(\alpha_1, \ldots, \alpha_p, \frac{x}{y}\right).$$

The left member equals $P(\alpha'_1)$ since

$$\alpha'_i = (\alpha_i - \alpha_1) + \alpha'_1, \qquad \frac{\xi}{\eta} = \left(\frac{x}{y} - \alpha_1\right) + \alpha'_1.$$

Hence

$$P(\alpha_1 - n) - P(\alpha_1) = 0$$

for every n. Hence α_1 does not occur in $P(\alpha_1)$, and κ is a polynomial in the differences of $x/y, \alpha_1, \ldots, \alpha_p$.

Let W be the weight of K and hence of the coefficient of y^ω. Then κ is of total degree W in the α's and of degree ω in x/y. Thus

$$\kappa = \Sigma c_i \left\{\text{product of } \omega \text{ differences like } \frac{x}{y} - \alpha_r\right\}$$
$$\cdot \{\text{product of } W - \omega \text{ differences like } \alpha_r - \alpha_s\}.$$

Hence

$$K = a_0{}^d \Sigma c_i \{\text{product of } \omega \text{ differences like } x - \alpha_r y\}$$
$$\cdot \{\text{product of } W - \omega \text{ differences like } \alpha_r - \alpha_s\}.$$

Next, for $x = -\eta,\ y = \xi,\ f$ becomes $F = A_0\xi^p + \ldots$ with a root $-1/\alpha_r$ corresponding to each root α_r of f. The function K for F is

$$A_0{}^d \Sigma c_i \left\{\text{product of } \omega \text{ differences like } \xi + \frac{1}{\alpha_r}\eta = \frac{(x - \alpha_r y)}{-\alpha_r}\right\}$$
$$\cdot \left\{\text{product of } W - \omega \text{ differences like } \frac{\alpha_r - \alpha_s}{\alpha_r \alpha_s}\right\}.$$

Using the value of A_0 in § 35, we see that the factor

$$(-1)^{pd} \alpha_1{}^d \ldots \alpha_p{}^d$$

must be cancelled by the $-\alpha_r$ and the $\alpha_r\alpha_s$ in the denominators.

Thus each term of the sum involves every root exactly d times. The signs agree since

$$dp=\omega+2(W-\omega),$$

as follows by counting the total number of α's.

Any covariant of degree d, order ω and weight W of

$$a_0(x-\alpha_1 y)\ \ldots\ (x-\alpha_p y)$$

equals the product of $a_0{}^d$ by a sum of products of constants and ω differences like $x-\alpha_r y$ and $W-\omega$ differences like $\alpha_r-\alpha_s$, such that every root occurs in exactly d factors of each product; moreover, the sum equals a symmetric function of the roots. Conversely, the product of $a_0{}^d$ by any such sum equals a rational integral covariant.

EXERCISES

1. $f=a_0x^3+3a_1x^2y+3a_2xy^2+a_3y^3$ has the covariant

$$K=a_0{}^2\sum_3(x-\alpha_1 y)^2(\alpha_2-\alpha_3)^2.$$

Show that the coefficient of x^2 in K equals $-18(a_0a_2-a_1{}^2)$. Why may we conclude that $K=-18H$, where H is the Hessian of f?

2. The same binary cubic has the covariant

$$a_0{}^2\sum_3(x-\alpha_1 y)(x-\alpha_2 y)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)=9H.$$

3. Every rational integral covariant of the binary quadratic f is a product of powers of f and its discriminant by a constant.

37. Covariant with a Given Leader S. If the seminvariant S has the factor a_0, and $S=a_0Q$, and if Q is the leader of a covariant K of f, then, since a_0 is the leader of f, S is the leader of the covariant fK. Hence it remains to consider only a seminvariant S not divisible by a_0. If S is of degree d and weight w,

$$S=a_0{}^d\Sigma c_i(\text{product of } w \text{ factors like } \alpha_r-\alpha_s),$$

where each product is of degree at most d in each root, and of degree exactly d in at least one root (§ 34). If each product is of degree d in every root, S is an invariant (§ 35) and hence is the required covariant. In the contrary case, let α_2, for example, enter to a degree less than d; we supply enough factors $x-\alpha_2 y$ to bring the degree in α_2 up to d. Then $a_0{}^d$

multiplied by the sum of the total products is a covariant with the leader S. For example,

$$a_0^2\sum_3(\alpha_2-\alpha_3)^2, \qquad a_0^2\sum_3(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)$$

are the leaders of the covariants in Exs. 1, 2, § 36, of the binary cubic. The present result should be compared with the theorem in § 25.

We may now give a new proof of the lemma in § 25 that $dp-2w\geqq 0$ for any seminvariant S of degree d and weight w of the binary p-ic. Whether S has the factor a_0 or not, the first term of the resulting covariant K is Sx^ω, where $\omega=dp-2w$. For, in each product in the above S, the roots $\alpha_1, \ldots, \alpha_p$ occur $2w$ times in all. In K each root occurs d times. Hence we inserted $dp-2w$ factors $x-\alpha y$ in deriving K from S.

38. Differential Operators Producing Covariants. Let the transformation

$$T: \qquad x=\alpha\xi+\beta\eta, \qquad y=\gamma\xi+\delta\eta, \qquad \Delta=\alpha\delta-\beta\gamma\neq 0$$

replace $f(x, y)$ by $\phi(\xi, \eta)$. Then

$$\frac{\partial\phi}{\partial\xi}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial\xi}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial\xi}=\alpha\frac{\partial f}{\partial x}+\gamma\frac{\partial f}{\partial y},$$

$$\frac{\partial\phi}{\partial\eta}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial\eta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial\eta}=\beta\frac{\partial f}{\partial x}+\delta\frac{\partial f}{\partial y}.$$

Solving, we get

$$\Delta\frac{\partial f}{\partial y}=\alpha\frac{\partial\phi}{\partial\eta}-\beta\frac{\partial\phi}{\partial\xi}, \qquad -\Delta\frac{\partial f}{\partial x}=\gamma\frac{\partial\phi}{\partial\eta}-\delta\frac{\partial\phi}{\partial\xi},$$

or $df=D\phi$, $d_1f=D_1\phi$, if we introduce the differential operators

$$d=\Delta\frac{\partial}{\partial y}, \qquad d_1=-\Delta\frac{\partial}{\partial x}, \qquad D=\alpha\frac{\partial}{\partial\eta}-\beta\frac{\partial}{\partial\xi}, \qquad D_1=\gamma\frac{\partial}{\partial\eta}-\delta\frac{\partial}{\partial\xi}.$$

As usual, write d^2d_1f for $d\{d(d_1f)\}$. Since the result of operating with d on df is the same as operating with D on the equal function $D\phi$ of ξ and η, we have $d^2f=D^2\phi$. Similarly,

$$\Sigma c_{rs}d^rd_1^sf=\Sigma c_{rs}D^rD_1^s\phi \qquad (r+s=\omega).$$

The right member is the result of operating on ϕ with the operator obtained by substituting D for $\partial/\partial\eta$ and D_1 for $-\partial/\partial\xi$ in

$$\Sigma c_{rs}\left(\frac{\partial}{\partial\eta}\right)^r\left(-\frac{\partial}{\partial\xi}\right)^s \qquad (r+s=\omega),$$

whose terms are partial derivatives of order ω. Hence, if the form

$$l(x, y)=\Sigma c_{rs}x^r y^s \qquad (r+s=\omega)$$

becomes $\lambda(\xi, \eta)$ under the transformation T, our right member is the result of operating on ϕ with $\lambda(\partial/\partial\eta, -\partial/\partial\xi)$. The left member is the result of operating on f with

$$l\left(\Delta\frac{\partial}{\partial y}, -\Delta\frac{\partial}{\partial x}\right)=\Delta^\omega l\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right).$$

Hence if T replaces the forms $f(x, y)$, $l(x, y)$ by $\phi(\xi, \eta)$, $\lambda(\xi, \eta)$, then

$$\left[\lambda\left(\frac{\partial}{\partial\eta}, -\frac{\partial}{\partial\xi}\right)\right]\phi(\xi, \eta)=\Delta^\omega\left[l\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)\right]f(x, y)$$

is a consequence of the equations for T, if ω is the order of $l(x, y)$.

Let f and l be covariants of indices m and n of one or more binary forms f_i with the coefficients $c_1, c_2, \ldots$. Under T let the transformed forms have the coefficients $C_1, C_2, \ldots$. Then

$$f(C;\ \xi, \eta)=\Delta^m f(c;\ x, y), \qquad l(C;\ \xi, \eta)=\Delta^n l(c;\ x, y).$$

But $\phi(\xi, \eta)=f(c;\ x, y)$, by the earlier notation. Hence

$$\phi(\xi, \eta)=\Delta^{-m}f(C;\ \xi, \eta), \qquad \lambda(\xi, \eta)=\Delta^{-n}l(C;\ \xi, \eta).$$

Inserting these into the formula of the theorem, and multiplying by Δ^{m+n}, we get

$$\left[l\left(C;\ \frac{\partial}{\partial\eta}, -\frac{\partial}{\partial\xi}\right)\right]f(C;\ \xi, \eta)=\Delta^{\omega+m+n}\left[l\left(c;\ \frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)\right]f(c;\ x,y).$$

The function in the right member is therefore a covariant of index $\omega+m+n$ of the f_i. We therefore have the theorem of Boole, one of the first known general theorems on covariants:

THEOREM. *If l and f are any covariants of a system of binary forms, we obtain a covariant (or invariant) of the system of forms by operating on f with the operator obtained from l by replacing x by $\partial/\partial y$ and y by $-\partial/\partial x$, i.e., $x^r y^s$ by $(-1)^s \partial^{r+s}/\partial y^r \partial x^s$.*

EXERCISES

1. Taking $l=f=ax^2+2bxy+cy^2$, obtain the invariant $4(ac-b^2)$ of f.

2. If $l=f$ is the binary quartic, the invariant is $2\cdot 4!\, I$ of § 31.

3. Using the binary quartic and its Hessian, obtain the invariant J.

4. Taking $l=a_0x^p+\ldots, f=b_0x^p+\ldots$, obtain their simultaneous invariant

$$\sum_{i=0}^{p}(-1)^i\binom{p}{i}a_ib_{p-i}.$$

If also $l\equiv f$, we have an invariant of f, which vanishes if p is odd. For $p=2$ and $p=4$, deduce the results in Exs. 1, 2.

5. A fundamental system of covariants of a quadratic and cubic

$$Q=Ax^2+2Bxy+Cy^2, \qquad f=ax^3+3bx^2y+3cxy^2+dy^3$$

is composed of 15 forms. We may take Q and its discriminant $AC-B^2$; f, its discriminant and Hessian h, given by (5) and (2) of § 8, the Jacobian J of f and H:

$$J=(a^2d-3abc+2b^3)x^3+3(abd+b^2c-2ac^2)x^2y$$
$$+3(2b^2d-acd-bc^2)xy^2+(3bcd-ad^2-2c^3)y^3;$$

the Jacobian of f and Q:

$$(Ab-Ba)x^3+(2Ac-Bb-Ca)x^2y+(Ad+Bc-2Cb)xy^2+(Bd-Cc)y^3;$$

the Jacobian of Q and h:

$$(As-Br)x^2+(At-Cr)xy+(Bt-Cs)y^2;$$

the result of operating on f with the operator obtained as in the theorem from $l=Q$:

$$L_1\equiv(aC+cA-2bB)x+(bC+dA-2cB)y;$$

the result of operating on Q with the operator obtained from L_1:

$$L_2=\{aBC-b(2B^2+AC)+3cAB-dA^2\}x$$
$$+\{aC^2-3bBC+c(AC+2B^2)-dAB\}y;$$

the result L_3, of operating on J with Q and the result L_4 of operating on Q with L_3 (so that L_3 and L_4 may be derived from L_1 and L_2 by replacing $a, \ldots, d$ by the corresponding coefficients of J); the intermediate invariant $At+Cr-2Bs$ of Q and h (§ 11); the resultant of Q and f:

$$a^2C^3-6abBC^2+6acC(2B^2-AC)+ad(6ABC-8B^3)+9b^2AC^2$$

$$-18bcABC+6bdA(2B^2-AC)+9c^2A^2C-6cdBA^2+d^2A^3;$$

the resultant of L_1 and L_4 (=resultant of L_2 and L_3), obtained at once as a determinant of order 2. Salmon, *Modern Higher Algebra*, § 198, gives geometrical interpretations. Hammond, *Amer. Jour. Math.*, vol. 8, obtains the syzygies between the 15 covariants.

PART III

SYMBOLIC NOTATION

The Notation and its Immediate Consequences, §§ 39–41

39. Introduction. The conditions that the binary cubic

(1) $$f=a_0x_1^3+3a_1x_1^2x_2+3a_2x_1x_2^2+a_3x_2^3$$

shall be a perfect cube

(2) $$(\alpha_1x_1+\alpha_2x_2)^3$$

are found by eliminating α_1 and α_2 between

(3) $$\alpha_1^3=a_0,\qquad \alpha_1^2\alpha_2=a_1,\qquad \alpha_1\alpha_2^2=a_2,\qquad \alpha_2^3=a_3,$$

and hence the conditions are

(4) $$a_0a_2=a_1^2,\qquad a_1a_3=a_2^2.$$

Thus only a very special form (1) is a perfect cube.

However, in a symbolic sense * any form (1) can be represented as a cube (2), in which α_1 and α_2 are now mere symbols such that

(3′) $$\alpha_1^3,\qquad \alpha_1^2\alpha_2,\qquad \alpha_1\alpha_2^2,\qquad \alpha_2^3$$

are given the interpretations (3), while any linear combination of these products, as $2\alpha_1^3-7\alpha_2^3$, is interpreted to be the corresponding combination of the a's, as $2a_0-7a_3$. But no interpretation is given to a polynomial in α_1, α_2, any one of whose terms is a product of more than three factors α, or fewer than three factors α. Thus the first relation (4) does not now follow from (3), since the expression $\alpha_1^4\alpha_2^2$ (formerly equal to both

* Due to Aronhold and Clebsch, but equivalent to the more complicated hyperdeterminants of Cayley.

a_0a_2 and a_1^2) is now excluded from consideration; likewise for $\alpha_1^2\alpha_2^4$ and the second relation (4).

In brief, the general binary cubic (1) may be represented in the symbolic form (2) since the products (3′) of the symbols α_1, α_2 are in effect independent quantities, in so far as we permit the use only of linear combinations of these products.

But we shall of course have need of other than linear functions of $a_0, \ldots, a_3$. To be able to express them symbolically, we represent f not merely by (2), but also in the symbolic forms

$$(5) \qquad (\beta_1x_1+\beta_2x_2)^3, \qquad (\gamma_1x_1+\gamma_2x_2)^3, \ldots,$$

so that

$$(6) \quad \beta_1^3=a_0, \quad \beta_1^2\beta_2=a_1, \quad \beta_1\beta_2^2=a_2, \quad \beta_2^3=a_3; \quad \gamma_1^3=a_0, \ldots.$$

Thus a_0a_2 is represented by either $\alpha_1^3\beta_1\beta_2^2$ or $\beta_1^3\alpha_1\alpha_2^2$, while neither of them is identical with the representation $\alpha_1^2\alpha_2\beta_1^2\beta_2$ of a_1^2. Hence

$$a_0a_2-a_1^2=\tfrac{1}{2}(\alpha_1^3\beta_1\beta_2^2+\beta_1^3\alpha_1\alpha_2^2-2\alpha_1^2\alpha_2\beta_1^2\beta_2)$$
$$=\tfrac{1}{2}\alpha_1\beta_1(\alpha_1\beta_2-\alpha_2\beta_1)^2.$$

We shall verify that this expression is a seminvariant of f. If

$$x_1=X_1+tX_2, \qquad x_2=X_2,$$

then f becomes $F=A_0X_1^3+\ldots$, where

$$A_0=a_0, \quad A_1=a_1+ta_0, \quad A_2=a_2+2ta_1+t^2a_0,$$
$$A_3=a_3+3ta_2+3t^2a_1+t^3a_0.$$

Hence, by (3),

$$F=(\alpha_1X_1+\alpha'_2X_2)^3, \qquad \alpha'_2=\alpha_2+t\alpha_1.$$

Similarly, the transform of (5_1) is

$$(\beta_1X_1+\beta'_2X_2)^3, \qquad \beta'_2=\beta_2+t\beta_1.$$

Hence we obtain the desired result

$$A_0A_2-A_1^2=\tfrac{1}{2}\alpha_1\beta_1(\alpha_1\beta'_2-\alpha'_2\beta_1)^2$$
$$=\tfrac{1}{2}\alpha_1\beta_1(\alpha_1\beta_2-\alpha_2\beta_1)^2=a_0a_2-a_1^2.$$

40. General Notations. The binary n-ic

$$f=a_0x_1^n+na_1x_1^{n-1}x_2+\ldots+\binom{n}{k}a_kx_1^{n-k}x_2^k+\ldots+a_nx_2^n$$

is represented symbolically as $\alpha_x^n=\beta_x^n=\ldots$, where

$$\alpha_x=\alpha_1x_1+\alpha_2x_2, \qquad \beta_x=\beta_1x_1+\beta_2x_2, \ \ldots,$$

$$\alpha_1^n=a_0, \quad \alpha_1^{n-1}\alpha_2=a_1, \ \ldots, \quad \alpha_1^{n-k}\alpha_2^k=a_k, \quad \ldots,$$

$$\alpha_2^n=a_n; \qquad \beta_1^n=a_0, \ \ldots.$$

A product involving fewer than n or more than n factors α_1, α_2 is not employed except, of course, as a component of a product of n such factors.

The general binary linear transformation is denoted by

$$T: \qquad x_1=\xi_1X_1+\eta_1X_2, \qquad x_2=\xi_2X_1+\eta_2X_2, \qquad (\xi\eta)\neq 0,$$

where $(\xi\eta)=\xi_1\eta_2-\xi_2\eta_1$. It is an important principle of computation, verified for a special case at the end of § 39, that T transforms α_x^n into the nth power of the linear function

$$(\alpha_1\xi_1+\alpha_2\xi_2)X_1+(\alpha_1\eta_1+\alpha_2\eta_2)X_2=\alpha_\xi X_1+\alpha_\eta X_2,$$

which is the transform of α_x by T. Further,

$$(1) \qquad \begin{vmatrix} \alpha_\xi & \alpha_\eta \\ \beta_\xi & \beta_\eta \end{vmatrix}=\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}\cdot\begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix}=(\alpha\beta)(\xi\eta),$$

where $(\alpha\beta)=\alpha_1\beta_2-\alpha_2\beta_1=-(\beta\alpha)$. Thus

$$(\alpha_\xi\beta_\eta-\alpha_\eta\beta_\xi)^n=(\xi\eta)^n(\alpha\beta)^n,$$

so that $(\alpha\beta)^n$ is an invariant of $\alpha_x^n=\beta_x^n$ of index n. Since $(\beta\alpha)^n$ represents the same invariant, the invariant is identically zero if n is odd.

EXERCISES

1. $(\alpha\beta)^2$ is the invariant $2(a_0a_2-a_1^2)$ of $\alpha_x^2=\beta_x^2$.
2. $(\alpha\beta)^4$ is the invariant $2I$ of $\alpha_x^4=\beta_x^4$ (§ 31).
3. $(\alpha\beta)^2(\beta\gamma)^2(\gamma\alpha)^2$ is the invariant $6J$ of $\alpha_x^4=\beta_x^4=\gamma_x^4$ (§ 31).
4. The Jacobian of α_x^m and β_x^n is

$$\begin{vmatrix} m\alpha_x^{m-1}\alpha_1 & m\alpha_x^{m-1}\alpha_2 \\ n\beta_x^{n-1}\beta_1 & n\beta_x^{n-1}\beta_2 \end{vmatrix}=mn(\alpha\beta)\alpha_x^{m-1}\beta_x^{n-1}.$$

5. The quotient of the Hessian of $\alpha_x^n = \beta_x^n$ by $n^2(n-1)^2$ equals

$$\begin{vmatrix} \alpha_x^{n-2}\alpha_1^2 & \alpha_x^{n-2}\alpha_1\alpha_2 \\ \beta_x^{n-2}\beta_1\beta_2 & \beta_x^{n-2}\beta_2^2 \end{vmatrix} = \begin{vmatrix} \beta_x^{n-2}\beta_1^2 & \beta_x^{n-2}\beta_1\beta_2 \\ \alpha_x^{n-2}\alpha_1\alpha_2 & \alpha_x^{n-2}\alpha_2^2 \end{vmatrix},$$

one-half of the sum of which equals $\frac{1}{2}\alpha_x^{n-2}\beta_x^{n-2}(\alpha\beta)^2$.

6. $$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_x & \beta_x & \gamma_x \end{vmatrix} = (\alpha\beta)\gamma_x + (\beta\gamma)\alpha_x + (\gamma\alpha)\beta_x \equiv 0.$$

41. Evident Covariants. We obtain a covariant K of

$$f = \alpha_x^n = \beta_x^n = \ldots$$

by taking a product of ω factors of type α_x and λ factors of type $(\alpha\beta)$, such that α occurs in exactly n factors, β in exactly n factors, etc. On the one hand, the product can be interpreted as a polynomial in $a_0, \ldots, a_n, x_1, x_2$. On the other hand, the product is a covariant of index λ of f, since, by (1), § 40,

$$(AB)^r(AC)^s(BC)^t \ldots A_X^a B_X^b C_X^c \ldots$$
$$= (\xi\eta)^\lambda(\alpha\beta)^r(\alpha\gamma)^s(\beta\gamma)^t \ldots \alpha_x^a\beta_x^b\gamma_x^c \ldots,$$

if $\lambda = r+s+t+\ldots$ and

$$A_X = A_1X_1 + A_2X_2, \quad A_1 = \alpha_\xi, \quad A_2 = \alpha_\eta, \quad (AB) = A_1B_2 - A_2B_1,$$

etc. The total degree of the right member in the α's, β's, . . . is $2\lambda + \omega = nd$, if d is the number of distinct pairs of symbols α_1, α_2; β_1, β_2; . . . in the product. Evidently d is the degree of K in $a_0, a_1, \ldots$, and ω is its order in x_1, x_2.

Any linear combination of such products with the same ω and λ, and hence same d, is a covariant of order ω, index λ and degree d of f.

EXERCISES

1. $(\alpha\beta)(\alpha\gamma)\alpha_x^3\beta_x^4\gamma_x^4$ and $(\alpha\beta)^2(\alpha\gamma)\alpha_x^2\beta_x^3\gamma_x^4$ are covariants of $\alpha_x^5 = \beta_x^5 = \gamma_x^5$.
2. $(\alpha\beta)^r\alpha_x^{n-r}\beta_x^{m-r}$ is a covariant of α_x^n, β_x^m.
3. If $m = n$, $\beta_x^n = \alpha_x^n$ and r is odd, the last covariant is identically zero.
4. $a_0x_1^2 + 2a_1x_1x^2 + a_2x_2^2$ and $b_0x_1^2 + 2b_1x_1x_2 + b_2x_2^2$ have the invariant

$$(\alpha\beta)^2 = a_0b_2 - 2a_1b_1 + a_2b_0.$$

COVARIANTS AS FUNCTIONS OF TWO SYMBOLIC TYPES, §§ 42–45

42. Any Covariant is a Polynomial in the α_x, $(\alpha\beta)$. This fundamental theorem, due to Clebsch, justifies the symbolic notation. It shows that any covariant can be expressed in a simple notation which reveals at sight the covariant property.

While a similar result was accomplished by expressing covariants in terms of the roots (§ 36), manipulations with symmetric functions of the roots are usually far more complex than those with our symbolic expressions.

The nature of the proof will be clearer if first made for a special case. The binary quadratic α_x^2 has the invariant

$$K = a_0a_2 - a_1^2$$

of index 2. Under transformation T of § 40, α_x^2 becomes

$$(\alpha_\xi X_1 + \alpha_\eta X_2)^2 = A_0X_1^2 + \ldots, \quad A_0 = \alpha_\xi^2, \quad A_1 = \alpha_\xi\alpha_\eta, \quad A_2 = \alpha_\eta^2.$$

Hence $A_0A_2 - A_1^2$ equals

$$\alpha_\xi^2\beta_\eta^2 - \alpha_\xi\beta_\xi\alpha_\eta\beta_\eta = (\xi\eta)^2K.$$

We operate on each member twice with

$$V = \frac{\partial^2}{\partial\xi_1\partial\eta_2} - \frac{\partial^2}{\partial\xi_2\partial\eta_1}, \tag{1}$$

and prove that we get $6(\alpha\beta)^2 = 12K$, so that K is expressed in the desired symbolic form. We have

$$(\xi\eta) = \xi_1\eta_2 - \xi_2\eta_1,$$

$$\frac{\partial}{\partial\eta_2}(\xi\eta)^2 = 2(\xi\eta)\xi_1, \qquad \frac{\partial^2}{\partial\xi_1\partial\eta_2}(\xi\eta)^2 = 2(\xi\eta) + 2\eta_2\xi_1,$$

$$\frac{\partial}{\partial\eta_1}(\xi\eta)^2 = -2(\xi\eta)\xi_2, \quad \frac{\partial^2}{\partial\xi_2\partial\eta_1}(\xi\eta)^2 = -2(\xi\eta) + 2\eta_1\xi_2,$$

$$V(\xi\eta)^2 = 6(\xi\eta), \qquad V^2(\xi\eta)^2 = 12,$$

since $V(\xi\eta) = 2$, by inspection. Next

$$V\alpha_\xi\beta_\eta = V(\alpha_1\xi_1 + \alpha_2\xi_2)(\beta_1\eta_1 + \beta_2\eta_2) = \alpha_1\beta_2 - \alpha_2\beta_1 = (\alpha\beta). \tag{2}$$

Hence

$$V\alpha_\xi^2\beta_\eta^2=4\alpha_\xi\beta_\eta(\alpha\beta),\quad V^2\alpha_\xi^2\beta_\eta^2=4(\alpha\beta)^2,$$

$$\begin{aligned}V\alpha_\xi\beta_\xi\alpha_\eta\beta_\eta&=\beta_\xi\alpha_\eta\cdot V\alpha_\xi\beta_\eta+\alpha_\xi\beta_\eta\cdot V\beta_\xi\alpha_\eta\\&=\beta_\xi\alpha_\eta(\alpha\beta)+\alpha_\xi\beta_\eta(\beta\alpha),\end{aligned}$$

$$V^2\alpha_\xi\beta_\xi\alpha_\eta\beta_\eta=(\beta\alpha)(\alpha\beta)+(\alpha\beta)(\beta\alpha)=-2(\alpha\beta)^2.$$

The difference of the expressions involving V^2 is $6(\alpha\beta)^2$. Hence if (1) operates twice on the equation preceding it, the result is

$$6(\alpha\beta)^2=12K,\qquad K=\tfrac{1}{2}(\alpha\beta)^2.$$

43. Lemma. $V^n(\xi\eta)^n=(n+1)(n!)^2$.

We have proved this for $n=1$ and $n=2$. If $n\geq 2$,

$$\frac{\partial}{\partial\eta_2}(\xi\eta)^n=n(\xi\eta)^{n-1}\xi_1,$$

$$\frac{\partial^2}{\partial\xi_1\partial\eta_2}(\xi\eta)^n=n(\xi\eta)^{n-1}+n(n-1)(\xi\eta)^{n-2}\eta_2\xi_1.$$

Similarly, or by interchanging subscripts 1 and 2, we get

$$\frac{\partial^2}{\partial\xi_2\partial\eta_1}(\xi\eta)^n=-n(\xi\eta)^{n-1}+n(n-1)(\xi\eta)^{n-2}\eta_1\xi_2.$$

Subtracting, we get

$$V(\xi\eta)^n=\{2n+n(n-1)\}(\xi\eta)^{n-1}=n(n+1)(\xi\eta)^{n-1}.$$

It follows by induction that, if r is a positive integer,

$$V^r(\xi\eta)^n=(n+1)\{n(n-1)\ \ldots\ (n-r+2)\}^2(n-r+1)(\xi\eta)^{n-r}.$$

The case $r=n$ yields the Lemma.

44. Lemma. *If the operator V is applied r times to a product of k factors of the type α_ξ and l factors of the type β_η, there results a sum of terms each containing $k-r$ factors α_ξ, $l-r$ factors β_η, and r factors $(\alpha\beta)$.*

The Lemma is a generalization of (2), § 42. To prove it, set

$$A=\alpha_\xi^{(1)}\alpha_\xi^{(2)}\ \ldots\ \alpha_\xi^{(k)},\qquad B=\beta_\eta^{(1)}\beta_\eta^{(2)}\ \ldots\ \beta_\eta^{(l)}.$$

Then

$$\frac{\partial^2 AB}{\partial\xi_1\partial\eta_2}=\sum_{s=1}^{k}\sum_{t=1}^{l}\alpha_1^{(s)}\beta_2^{(t)}\frac{A}{\alpha_\xi^{(s)}}\frac{B}{\beta_\eta^{(t)}},$$

$$\frac{\partial^2 AB}{\partial\xi_2\partial\eta_1}=\sum_{s=1}^{k}\sum_{t=1}^{l}\alpha_2^{(s)}\beta_1^{(t)}\frac{A}{\alpha_\xi^{(s)}}\frac{B}{\beta_\eta^{(t)}}.$$

Subtracting, we get

$$VAB=\sum_{s=1}^{k}\sum_{t=1}^{l}(\alpha^{(s)}\beta^{(t)})\frac{A}{\alpha_\xi^{(s)}}\frac{B}{\beta_\eta^{(t)}}.$$

Hence the lemma is true when $r=1$. It now follows at once by induction that

(1) V^rAB

$$=\Sigma\Sigma(\alpha^{(s_1)}\beta^{(t_1)})\;.\;.\;.\;(\alpha^{(s_r)}\beta^{(t_r)})\frac{A}{\alpha_\xi^{(s_1)}\;.\;.\;.\;\alpha_\xi^{(s_r)}}\frac{B}{\beta_\eta^{(t_1)}\;.\;.\;.\;\beta_\eta^{(t_r)}},$$

where the first summation extends over all of the $k(k-1)\;.\;.\;.\;(k-r+1)$ permutations $s_1,\;.\;.\;.\;,s_r$ of $1,\;.\;.\;.\;,k$ taken r at a time, and the second summation extends over all of the $l(l-1)\;.\;.\;.\;(l-r+1)$ permutations $t_1,\;.\;.\;.\;,t_r$ of $1,\;.\;.\;.\;,l$ taken r at a time.

Corollary. The terms of (1) coincide in sets of $r!$ and the number of formally distinct terms is

$$\frac{k!}{(k-r)!}\cdot\frac{l!}{(l-r)!}\cdot\frac{1}{r!}=\binom{k}{r}\binom{l}{r}\cdot r!.$$

For, we obtain the same product of determinantal factors if we rearrange $s_1,\;.\;.\;.\;,s_r$ and make the same rearrangement of $t_1,\;.\;.\;.\;,t_r$.

45. Proof of the Fundamental Theorem in § 42. Let K be a homogeneous covariant of order ω and index λ of the binary form f in § 40. By § 40, the general linear transformation replaces $f=\alpha_x^n$ by

$$\sum_{k=0}^{n}\binom{n}{k}A_kX_1^{n-k}X_2^k=(\alpha_\xi X_1+\alpha_\eta X_2)^n.$$

Hence

(1) $$A_k=\alpha_\xi^{n-k}\alpha_\eta^k \qquad (k=0,1,\;.\;.\;.\;,n).$$

By the covariance of K,

(2) $$K(A_0, \ldots, A_n;\ X_1, X_2) = (\xi\eta)^\lambda K(a_0, \ldots, a_n;\ x_1, x_2).$$

By (1) the left member equals

$$\sum_{i=0}^{\omega} \Sigma ABX_1^{\omega-i}X_2^{i},$$

in which the inner summation extends over various products AB, where A is a product of a constant and factors of type α_ξ, and B is a product of a constant and factors of type α_η. Let $x_1 = y_2$, and $x_2 = -y_1$. Then, by solving the equations of T, § 40,

$$X_1 = y_\eta/(\xi\eta), \qquad X_2 = -y_\xi/(\xi\eta).$$

Hence the equation (2) becomes

$$\sum_{i=0}^{\omega} \Sigma(-1)^i ABy_\eta^{\omega-i}y_\xi^{i} = (\xi\eta)^{\lambda+\omega}K.$$

Since the right member is of degree $\lambda+\omega$ in ξ_1, ξ_2, and of degree $\lambda+\omega$ in η_1, η_2, we infer that each term of the left member involves exactly $\lambda+\omega$ factors with subscript ξ and $\lambda+\omega$ factors with subscript η.

Operate with $V^{\lambda+\omega}$ on each member. By § 43, the right member becomes cK, where c is a numerical constant $\neq 0$. By § 44, the left member becomes a sum of products each of $\lambda+\omega$ determinantal factors of which ω are of type $(\alpha y) \equiv \alpha_x$, and hence λ of type $(\alpha\beta)$. The last is true also by the definiton of the index λ of K. Hence K equals a polynomial in the symbols of the types α_x, $(\alpha\beta)$.

To extend the proof to covariants of several binary forms α_x^n, γ_x^m, . . . , we employ, in addition to (1), $C_k = \gamma_\xi^{m-k}\gamma_\eta^k$, . . . and read α_ξ, γ_ξ, . . . for α_ξ in the above proof.

Finiteness of a Fundamental System of Covariants, §§ 46–51

46. Remarks on the Problem. It was shown in §§28–31 that a binary form f of order <5 has a finite fundamental system of rational integral covariants $K_1, \ldots, K_s$, such therefore that any rational integral covariant of f is a poly-

nomial in $K_1, \ldots, K_s$ with numerical coefficients. We shall now prove a like theorem for the covariants of any system of binary forms of any orders. The first proof was that by Gordan; it was based upon the symbolic notation and gave the means of actually constructing a fundamental system. Cayley had earlier come to the conclusion that the fundamental system for a binary quintic is infinite, after making a false assumption on the independence of the syzygies between the covariants. The proof reproduced here is one of those by Hilbert; it is merely an existence proof, giving no clue as to the actual covariants in a fundamental system.

47. Reduction of the Problem on Covariants to one on Invariants. We shall prove that the set of all covariants of the binary forms $f_1, \ldots, f_k$ is identical with the set of forms derived from the invariants I of $f_1, \ldots, f_k$ and $l \equiv xy'-x'y$ by replacing x' by x and y' by y in each I. It is here assumed (§ 15) that I is homogeneous in the coefficients of l and that the covariants are homogeneous in the variables.

Let the coefficients of the f's be $a, b, \ldots$, arranged in any sequence. Let $A, B, \ldots$ be the corresponding coefficients of the forms obtained by applying the transformation in § 5. The latter replaces l by $\xi\eta'-\xi'\eta$, where

$$\eta'=\alpha y'-\gamma x', \qquad \xi'=\delta x'-\beta y'.$$

Solving these, we get

$$\Delta x'=\alpha\xi'+\beta\eta', \qquad \Delta y'=\gamma\xi'+\delta\eta'.$$

Let $I(a, b, \ldots; x', y')$ be an invariant of l and the f's. Then

$$I(A, B, \ldots; \xi', \eta')=\Delta^{\lambda}I(a, b, \ldots; x', y').$$

Since I is homogeneous, of order ω, in x', y', the right member equals

$$\Delta^{\lambda-\omega}I(a, b, \ldots; \Delta x', \Delta y').$$

Hence we have the identity in ξ', η':

$$I(A, B, \ldots; \xi', \eta')\equiv\Delta^{\lambda-\omega}I(a, b, \ldots; \alpha\xi'+\beta\eta', \quad \gamma\xi'+\delta\eta').$$

Thus we may remove the accents on ξ', η'. Then, by our transformation,

$$I(A, B, \ldots ; \xi, \eta)=\Delta^{\lambda-\omega}I(a, b, \ldots ; x, y).$$

Hence $I(a, b, \ldots ; x, y)$ is a covariant of $f_1, \ldots, f_k$ of order ω and index $\lambda-\omega$.

The argument can be reversed. Note that the sum of the order and the index of a covariant is its weight (§ 22) and hence is not negative.

COROLLARY. A covariant of the binary form f has the annhilators in § 23.

For, an invariant of f and $xy'-x'y$ has the annihilators

$$\Omega-y'\frac{\partial}{\partial x'}, \qquad O-x'\frac{\partial}{\partial y'}.$$

48. Hilbert's Theorem. *Any set S of forms in $x_1, \ldots, x_n$ contains a finite number of forms $F_1, \ldots, F_k$ such that any form F of the set can be expressed as $F\equiv f_1F_1+\ldots+f_kF_k$, where $f_1, \ldots, f_k$ are forms in $x_1, \ldots, x_n$, but not necessarily in the set S.*

For $n=1$, S is composed of certain forms $c_1x^{e_1}$, $c_2x^{e_2}$, ... Let e_s be the least of the e's, and set $F_1=c_sx^{e_s}$. Then each form in S is the product of F_1 by a factor of the form cx^e, $e\geqq 0$. Thus the theorem holds when $n=1$.

To proceed by induction, let the theorem hold for every set of forms in $n-1$ variables. To prove it for the system S, we may assume, without real loss of generality,* that S contains a form F_0 of total order r in which the coefficient of $x_n{}^r$ is not zero. Let F be any form of the set S. By division we have $F=F_0P+R$, where R is a form whose order in x_n

* Let F be a form in S not identically zero and let the linear transformation

$$x_i=c_{i1}y_1+c_{i2}y_2+\ldots+c_{in}y_n \qquad (i=1, \ldots, n)$$

replace $F(x_1, \ldots, x_n)$ by $K(y_1, \ldots, y_n)$. In the latter the coefficient of the term involving only y_n is obtained from F by setting $x_i=c_{in}$ and hence is $F(c_{1n}, c_{2n}, \ldots, c_{nn})$, which is not zero for suitably chosen c's (Weber's *Algebra*, vol. I, p. 457; second edition, p. 147). But our theorem will be true for S if proved true for the set of forms K.

is $<r$. In R we segregate the terms whose order in x_n is exactly $r-1$, and have

$$F=F_0P+Mx_n^{r-1}+N,$$

where M is a form in $x_1, \ldots, x_{n-1}$, while N is a form in $x_1, \ldots, x_n$ whose order in x_n is $\leqq r-2$. Each F uniquely determines an M.

For the definite set of forms M in $n-1$ variables the theorem is true by hypothesis. Hence there exists a finite number of the M's, say $M_1, \ldots, M_l$ (derived from $F_1, \ldots, F_l$), such that any M can be expressed as

$$M=f_1M_1+\ldots+f_lM_l,$$

where the f's are forms in $x_1, \ldots, x_{n-1}$. Then

$$F=F_0P+N+x_n^{r-1}\sum_{i=1}^{l} f_iM_i, \quad x_n^{r-1}M_i=F_i-F_0P_i-N_i,$$

$$F=F_0P'+\sum_{i=1} f_iF_i+R', \quad P'\equiv P-\Sigma f_iP_i, \quad R'\equiv N-\Sigma f_iN_i.$$

Each exponent of x_n in R' is $\leqq r-2$. We segregate its terms in which this exponent is exactly $r-2$ and have

$$F=F_0P'+\sum_{i=1}^{l} f_iF_i+M'x_n^{r-2}+N',$$

where M' is a form in $x_1, \ldots, x_{n-1}$, and N' a form in $x_1, \ldots, x_n$ whose order in x_n is $\leqq r-3$.

The theorem is applicable to the set of forms M', so that each is a linear combination of $M'_1, \ldots, M'_m$, corresponding to $F_{l+1}, \ldots, F_{l+m}$, say. As before, F differs from a linear combination of $F_0, \ldots, F_{l+m}$ by

$$M''x_n^{r-3}+N'',$$

where M'' is a form in $x_1, \ldots, x_{n-1}$ and N'' is a form whose order in x_n is $\leqq r-4$. Proceeding in this manner, we see that F differs from a linear combination of $F_0, \ldots, F_t$ by a form $\bar{R}$ in $x_1, \ldots, x_{n-1}$. One more step leads to the theorem.

49. Finiteness of a Fundamental System of Invariants. Consider the set of all invariants of the binary forms $f_1, \ldots, f_d$,

homogeneous in the coefficients of each form separately. By the preceding theorem, there is a finite number of these invariants $I_1, \ldots, I_m$ in terms of which any one of the invariants I is expressible linearly:

$$I = E_1 I_1 + \ldots + E_m I_m, \tag{1}$$

where E_j is not necessarily an invariant, but is a polynomial homogeneous in the coefficients of each f_i separately.

Let $a_1, a_2, \ldots$ be the coefficients in any order of $f_1, \ldots, f_d$. Let $A_1, A_2, \ldots$ be the coefficients in the same order of the forms obtained from them by applying a linear transformation of determinant $(\xi\eta)$. We may write

$$I(A) = (\xi\eta)^\lambda I(a), \qquad I_j(A) = (\xi\eta)^{\lambda_j} I_j(a), \qquad E_j(A) = G_j,$$

where G_j is a function of the a's, ξ's, η's. From the identity (1) in the a's, we obtain an identity by replacing the a's by the A's. Hence

$$(\xi\eta)^\lambda I = \sum_{j=1}^{m} G_j (\xi\eta)^{\lambda_j} I_j,$$

in which the arguments of the I's are a's. Thus G_j is of order $\lambda - \lambda_j$ in ξ_1, ξ_2 and of order $\lambda - \lambda_j$ in η_1, η_2. Operate on each member by V^λ. By § 43, the left member becomes

$$(\lambda+1)(\lambda!)^2 I.$$

By the formula to be proved in § 50, the right member becomes

$$\sum_{j=1}^{m} I_j \{C_0 (\xi\eta)^{\lambda_j - \lambda} G_j + C_1 (\xi\eta)^{\lambda_j - \lambda + 1} V G_j + \ldots + C_\lambda (\xi\eta)^{\lambda_j} V^\lambda G_j\},$$

where the C's are numerical constants. Since G_j is of order $\nu \equiv \lambda - \lambda_j \geqq 0$ in ξ_1, ξ_2 and of order ν in η_1, η_2,

$$V^{\nu+1} G_j = 0, \qquad V^{\nu+2} G_j = 0, \qquad \ldots, \qquad V^\lambda G_j = 0.$$

Also $C_0, C_1, \ldots, C_{\nu-1}$ are zero since they multiply powers of $(\xi\eta)$ whose exponents $-\nu, -\nu+1, \ldots, \lambda_j - \lambda + \nu - 1 = -1$ are negative. Hence

$$(\lambda+1)(\lambda!)^2 I = \sum_{j=1}^{m} I_j C_\nu V^\nu G_j.$$

The torm obtained from $f_i=\alpha_x^n$ by our linear transformation has the coefficients (1), § 45. The polynomial G_j in these coefficients is therefore a sum of terms each a product of a constant by ν factors of type α_ξ and ν factors of type α_η. Hence, by § 44, $V^\nu G_j$ is a polynomial in the determinantal factors $(\alpha\beta)$ and is consequently an invariant of the forms f_i. Thus

$$I=\sum_{j=1}^{m} I_j I'_j,$$

where I'_j is an invariant. Then, by (1),

$$I'_j=\sum_{k=1}^{m} e_{jk}I_k, \qquad I=\sum_{j,\,k=1}^{m} e_{jk}I_jI_k.$$

By repeating the former process on this I, we get

$$I=\sum_{j,\,k=1}^{m} I''_{jk}I_jI_k,$$

where the I'' are invariants of the forms f_i. Since there is a reduction of degree at each step, we ultimately obtain an expression for I as a polynomial in $I_1, \ldots, I_m$ with numerical coefficients.

50. Lemma. *If $D=\xi_1\eta_2-\xi_2\eta_1$, and P is homogeneous (of order λ) in ξ_1, ξ_2, and homogeneous (of order μ) in η_1, η_2, then*

$$V^mD^nP=\sum_{r=0}^{m} C_rD^{n-m+r}V^rP, \tag{1}$$

where $C_0, \ldots, C_m$ are constants.

First, we have

$$VDP=P+\eta_2\frac{\partial P}{\partial \eta_2}+\xi_1\frac{\partial P}{\partial \xi_1}+D\frac{\partial^2 P}{\partial \xi_1\partial \eta_2}$$
$$-\left(-P-\xi_2\frac{\partial P}{\partial \xi_2}-\eta_1\frac{\partial P}{\partial \eta_1}+D\frac{\partial^2 P}{\partial \xi_2\partial \eta_1}\right)=(2+\lambda+\mu)P+DVP,$$

by Euler's theorem for homogeneous functions (§ 24). If P is replaced by $D^{n-1}P$, so that λ and μ are increased by $n-1$, we get

$$VD^nP=(\lambda+\mu+2n)D^{n-1}P+DVD^{n-1}P.$$

Using this as a recursion formula, we get

$$VD^nP=\{n(\lambda+\mu)+n(n+1)\}D^{n-1}P+D^nVP,$$

which reduces to the result in § 43 if $P=1$, whence $\lambda=\mu=0$. Hence (1) holds when $m=1$. To proceed by induction from m to $m+1$, apply V to (1). Thus

$$V^{m+1}D^nP=\sum_{r=0}^{m} C_rV(D^{n-m+r}V^rP).$$

In the result for VD^nP, replace n by $n-m+r$ and P by V^rP, and therefore diminish λ and μ by r. We get

$$V(D^{n-m+r}V^rP)=t_rD^{n-m+r-1}V^rP+D^{n-m+r}V^{r+1}P,$$

where

$$t_r=(n-m+r)(\lambda+\mu-r+n-m+1).$$

Hence, changing $r+1$ to r in the second summand, we get

$$V^{m+1}D^nP=\sum_{r=0}^{m+1}(C_rt_r+C_{r-1})D^{n-m+r-1}V^rP,$$

with $C_{m+1}=0$, $C_{-1}=0$. Thus (1) is true for every m.

51. Finiteness of Syzygies. Let $I_1, \ldots, I_m$ be a fundamental system of invariants of the binary forms $f_1, \ldots, f_d$. Let $S(z_1, \ldots, z_m)$ be a polynomial with numerical coefficients such that $S(I_1, \ldots, I_m)$, when expressed as a function of the coefficients c of the f's, is identically zero in the c's. Then $S(I)=0$ is a syzygy between the invariants.

By means of a new variable z_{m+1}, construct the homogeneous form $S'(z_1, \ldots, z_{m+1})$ corresponding to S. By § 48, the forms S' are expressible linearly in terms of a finite number $S'_1, \ldots, S'_k$ of them. Take $z_{m+1}=1$. Thus

$$S=C_1S_1+\ldots+C_kS_k, \tag{1}$$

where $C_1, \ldots, C_k$ are polynomials in $z_1, \ldots, z_m$. Take $z_1=I_1, \ldots, z_m=I_m$. Hence there is a finite number of syzygies $S_1=0, \ldots, S_k=0$, such that any syzygy $S=0$ implies a relation (1) in which $C_1, \ldots, C_k$ are invariants. In particular, every syzygy is a consequence of $S_1=0, \ldots, S_k=0$.

52. Transvectants. Any two binary forms

$$f=\alpha_x^k, \qquad \phi=\beta_x^l$$

have the covariant

$$(1) \qquad (f,\ \phi)^r=(\alpha\beta)^r\alpha_x^{k-r}\beta_x^{l-r},$$

called the rth transvectant (Ueberschiebung) of f and ϕ, and due to Cayley. It is their product if $r=0$, their Jacobian if $r=1$, and their Hessian if $f\equiv\phi$ and $r=2$, provided numerical factors are ignored (Exs. 4, 5, § 40).

It may be obtained by differentiation and without the use of the symbolic notation. In fact, a special case of (1), § 44, is

$$V^r\alpha_\xi^k\beta_\eta^l=\frac{k!}{(k-r)!}\,\frac{l!}{(l-r)!}(\alpha\beta)^r\alpha_\xi^{k-r}\beta_\eta^{l-r},$$

so that if f is of order k and ϕ of order l,

$$(2) \qquad (f(\xi),\ \phi(\xi))^r=\frac{(k-r)!}{k!}\,\frac{(l-r)!}{l!}[V^rf(\xi)\phi(\eta)]_{\eta=\xi}.$$

After $f(\xi_1,\ \xi_2)\cdot\phi(\eta_1,\ \eta_2)$ is operated on by V^r, we set $\eta_1=\xi_1$, $\eta_2=\xi_2$.

For example, let $f(\xi)=\alpha_\xi\beta_\xi$, $\phi(\xi)=\gamma_\xi^3$, $P=\alpha_\xi\beta_\xi\gamma_\eta^3$. Then

$$\frac{\partial^2P}{\partial\xi_1\partial\eta_2}=3(\alpha_\xi\beta_1+\alpha_1\beta_\xi)\gamma_\eta^2\gamma_2,\qquad \frac{\partial^2P}{\partial\xi_2\partial\eta_1}=3(\alpha_\xi\beta_2+\alpha_2\beta_\xi)\gamma_\eta^2\gamma_1.$$

The difference is VP. Taking $\eta_1=\xi_1$, $\eta_2=\xi_2$, we get

$$3\{\alpha_\xi(\beta_1\gamma_2-\beta_2\gamma_1)+\beta_\xi(\alpha_1\gamma_2-\alpha_2\gamma_1)\}\gamma_\xi^2.$$

The numerical factor in (2) is here 1/6. Hence

$$(3) \qquad (\alpha_\xi\beta_\xi,\ \gamma_\xi^3)^1=\tfrac{1}{2}(\beta\gamma)\alpha_\xi\gamma_\xi^2+\tfrac{1}{2}(\alpha\gamma)\beta_\xi\gamma_\xi^2.$$

In general, consider the two forms

$$f=\alpha_\xi^{(1)}\alpha_\xi^{(2)}\ .\ .\ .\ \alpha_\xi^{(k)}, \qquad \phi=\beta_\xi^{(1)}\beta_\xi^{(2)}\ .\ .\ .\ \beta_\xi^{(l)}.$$

Then by (1), § 44, and the Corollary, and by (2),

$$(4) \qquad (f,\ \phi)^r=\frac{1}{r!\binom{k}{r}\binom{l}{r}}\sum\frac{(\alpha^{(1)}\beta^{(1)})\ .\ .\ .\ (\alpha^{(r)}\beta^{(r)})\,f\phi}{\alpha_\xi^{(1)}\ .\ .\ .\ \alpha_\xi^{(r)}\beta_\xi^{(1)}\ .\ .\ .\ \beta_\xi^{(r)}},$$

where the summation extends over all the combinations of the

α's r at a time, and over all the permutations of the β's r at a time. Thus the number of terms in the sum is the reciprocal of the factor preceding Σ.

If the α's are identified and also the β's, (4) becomes (1). If $k=2$, $l=3$, $r=1$, we have one-sixth of a sum of six terms; then if the β's are identified we have two sets of three equal terms and obtain (3).

Since V is a differential operator, (2) gives

$$(5) \qquad (\Sigma c_i f_i,\ \Sigma k_j \phi_j)^r = \Sigma\Sigma c_i k_j (f_i,\ \phi_j)^r.$$

Apolarity; Rational Curves, §§ 53–57

53. Binary Forms Apolar to a Given Form. Two binary quadratic forms are called apolar if their lineo-linear invariant is zero; then they are harmonic (Ex. 3, § 11). In general, the binary forms

$$f=\alpha_x^n=\sum_{i=0}^{n}\binom{n}{i}a_i x_1^{n-i}x_2^i, \qquad \phi=\beta_x^n=\sum_{i=0}^{n}\binom{n}{i}b_i x_1^{n-i}x_2^i,$$

of the same order, are called apolar if

$$(1) \qquad (\alpha\beta)^n=\sum_{i=0}^{n}(-1)^i\binom{n}{i}a_i b_{n-i}=0.$$

In particular, f is apolar to itself if n is odd (Ex. 4, § 38).

Let the actual linear factors of ϕ be $\beta_x^{(1)}, \ldots, \beta_x^{(n)}$. By (1), (4), § 52,

$$(\alpha\beta)^n=(\alpha_x^n,\ \beta_x^{(1)} \ldots \beta_x^{(n)})^n=(\alpha\beta^{(1)}) \ldots (\alpha\beta^{(n)}).$$

But $\beta_x^{(r)}$ vanishes if x_1 and x_2 equal respectively

$$y_1^{(r)}=\beta_2^{(r)}, \qquad y_2^{(r)}=-\beta_1^{(r)}.$$

Thus

$$(\alpha\beta^{(r)})=\alpha_1 y_1^{(r)}+\alpha_2 y_2^{(r)}=\alpha_{y^{(r)}}.$$

Hence if ϕ vanishes for $x_1=y_1^{(r)}$, $x_2=y_2^{(r)}$ $(r=1, \ldots, n)$, it is apolar to f if and only if

$$\alpha_{y^{(1)}}\,\alpha_{y^{(2)}} \ldots \alpha_{y^{(n)}}=0.$$

Thus f is apolar to an actual nth power $(y_2x_1-y_1x_2)^n$ if and only if $\alpha_y^n=0$, i.e., if y_1, y_2 is a pair of values for which $f=0$.

If no two of the actual linear factors l_i of f are proportional, f is apolar to n actual nth powers l_i^n and these are readily seen to be linearly independent. Then their linear combinations give all the forms apolar to f. For, if f is apolar to $\phi_1, \ldots, \phi_n$, it is apolar to $k_1\phi_1+\ldots+k_n\phi_n$, where $k_1, \ldots, k_n$ are constants, since, by (5), § 52,

$$(f, k_1\phi_1+\ldots+k_n\phi_n)^n = k_1(f, \phi_1)^n+\ldots+k_n(f, \phi_n)^n = 0.$$

Moreover, f is not apolar to $n+1$ linearly independent forms

$$\phi_1, \phi_2, \ldots, \phi_{n+1}.$$

For, if so, we have $n+1$ equations like (1), in which the determinant of the coefficients of $a_0, \ldots, a_n$ is therefore zero. But this implies a linear relation between the ϕ's. *If f is the product of n distinct linear factors l_i, a form ϕ can be represented as a linear combination of $l_1^n, \ldots, l_n^n$ if and only if ϕ is apolar to f.* In particular, if r and s are the distinct roots of $f \equiv ax^2+2bx+c=0$, the only quadratics harmonic to f are $g(x-r)^2+h(x-s)^2$.

In case $l_1, \ldots, l_r$ are identical, while $l_1 \neq l_i (i>r)$, we may replace $l_1^n, \ldots, l_r^n$ in the above discussion by $l_1^n, l_1^{n-1}\lambda, \ldots, l_1^{n-r+1}\lambda^{r-1}$, where λ is any linear function of x_1 and x_2 which is linearly independent of l_1. In fact, after a linear transformation of variables, we may set $l_1=x_2$, $\lambda=x_1$. Then the above r forms have the factor x_2^{n-r+1} and hence are of type ϕ with $b_i=0 (i \leqq n-r)$. Also, f now has the factor x_2^r, so that $a_i=0 (i<r)$. Hence every term of (1) is zero.

For example, $f=x_1^2x_2(x_1-x_2)^2$ is apolar to

$$x_1^5,\ x_1^4x_2;\quad x_2^5;\quad (x_1-x_2)^5,\ (x_1-x_2)^4x_1,$$

which give five linearly independent quintics.

In general, when there are multiple factors of f, the n forms apolar to f obtained above can be proved to be linearly independent. This fact is not presupposed in what follows.

54. Binary Forms Apolar to Several Given Forms. From the list of the given forms we may drop any one linearly de-

pendent on the others, since a form apolar to several forms is apolar to any linear combination of them. In the resulting linearly independent forms

$$f_r=\sum_{i=0}^{n}\binom{n}{i}a_{ir}x_1^{n-i}x_2^{i}\qquad (r=1,\ .\ .\ .\ ,g),$$

the g-rowed determinants in the rectangular array of the coefficients are not all zero. For, if so, there are solutions $k_1, \ldots, k_g$, not all zero, of

$$k_1a_{i1}+k_2a_{i2}+\ldots+k_ga_{ig}=0\quad (i=0,1,\ .\ .\ .\ ,n),$$

which would give, contrary to hypothesis, the identity

$$k_1f_1+k_2f_2+\ldots+k_gf_g\equiv 0.$$

If $b_0x_1^n+\ldots$ is apolar to each f_r, then

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}a_{ir}b_{n-i}=0\qquad (r=1,\ .\ .\ .\ ,g).$$

These determine g of the b's as linear functions of the remaining b's, which are arbitrary. Hence there are exactly $n+1-g$ linearly independent forms apolar to each of the g given linearly independent forms.

In particular, apart from a constant factor, there is a single form apolar to each of n given linearly independent forms of order n.

Consider three binary cubic forms

$$f_1=\alpha_x^3=a_0x_1^3+3a_1x_1^2x_2+3a_2x_1x_2^2+a_3x_2^3,$$

$$f_2=\beta_x^3=b_0x_1^3+3b_1x_1^2x_2+3b_2x_1x_2^2+b_3x_2^3,$$

$$f_3=\gamma_x^3=c_0x_1^3+3c_1x_1^2x_2+3c_2x_1x_2^2+c_3x_2^3.$$

Each is apolar to the cubic form

$$\phi=(\alpha\beta)(\alpha\gamma)(\beta\gamma)\alpha_x\beta_x\gamma_x.$$

For, by (4), § 52, and the removal of a constant factor by (5),

$$(\phi,\ \delta_x^3)^3=(\alpha\beta)(\alpha\gamma)(\beta\gamma)(\alpha\delta)(\beta\delta)(\gamma\delta),$$

which is changed in sign if δ is interchanged with α, β, or γ,

and hence is zero if δ_x^3 is one of the f_i. Hence each f_i is apolar to ϕ. Now

$$(\alpha\beta)(\alpha\gamma)(\beta\gamma)=\begin{vmatrix}\alpha_1^2 & \alpha_1\alpha_2 & \alpha_2^2\\ \beta_1^2 & \beta_1\beta_2 & \beta_2^2\\ \gamma_1^2 & \gamma_1\gamma_2 & \gamma_2^2\end{vmatrix}.$$

In fact, the determinant vanishes if $(\alpha\beta)=0$ as may be seen by setting $\beta_1=c\alpha_1$, $\beta_2=c\alpha_2$. Moreover, the two members are of total degree six and the diagonal term of the determinant equals the product of the first terms $\alpha_1\beta_2$, etc., on the left.

Since $\alpha_1^2\alpha_x=\alpha_1^3x_1+\alpha_1^2\alpha_2x_2=a_0x_1+a_1x_2$, etc., we find, by multiplying the members of the last equation by $\alpha_x\beta_x\gamma_x$,

$$\phi=\begin{vmatrix}a_0x_1+a_1x_2 & a_1x_1+a_2x_2 & a_2x_1+a_3x_2\\ b_0x_1+b_1x_2 & b_1x_1+b_2x_2 & b_2x_1+b_3x_2\\ c_0x_1+c_1x_2 & c_1x_1+c_2x_2 & c_2x_1+c_3x_2\end{vmatrix}$$

$$=[012]x_1^3+[013]x_1^2x_2+[023]x_1x_2^2+[123]x_2^3,$$

where

$$[ijk]=\begin{vmatrix}a_i & a_j & a_k\\ b_i & b_j & b_k\\ c_i & c_j & c_k\end{vmatrix}.$$

If ϕ is identically zero, the four three-rowed determinants in the rectangular array of the coefficients of f_1, f_2, f_3 are all zero, and the f's are linearly dependent.

Apart from a constant factor, ϕ is the unique form apolar to three linearly independent cubic forms f_1, f_2, f_3.

The extension to n binary n-ics is readily made.

55. Rational Plane Cubic Curves. The homogeneous coördinates ξ, η, ζ of a point on such a curve are cubic functions of a parameter t. We may take $t=x_1/x_2$ and write

$$\rho\xi=f_1, \qquad \rho\eta=f_2, \qquad \rho\zeta=f_3,$$

where ρ is a factor of proportionality and the f's are the cubic forms in § 54.

We may assume that the f's are linearly independent, since otherwise all of the points (ξ, η, ζ) would lie on a straight line.

There is a unique cubic form ϕ apolar to f_1, f_2, f_3 (§ 54). This cubic form, denoted by $\phi=\phi_x^3$, is fundamental in the theory of the cubic curve.

Three points determined by the pairs of parameters x_1, x_2; y_1, y_2; *and* z_1, z_2, *are collinear if and only if*

(1) $$\phi_x\phi_y\phi_z=0.$$

For, if the three points lie on the straight line

(2) $$l\xi+m\eta+n\zeta=0,$$

the three pairs of parameters are pairs of values for which

(3) $$C(x_1, x_2)\equiv lf_1+mf_2+nf_3=0.$$

Since C is apolar to ϕ, (1) follows from the first italicized theorem in § 53. Conversely, (1) implies that the cubic C which vanishes for the three pairs of parameters is apolar to ϕ and hence (§ 53) is a linear combination of f_1, f_2, f_3, say (3); the corresponding three points lie on the straight line (2).

Since (2) meets the curve in three points the ratios x_1/x_2 of whose parameters are the roots of (3), the curve is of the third order.

We restrict attention to the case in which the actual linear factors α_x, β_x, γ_x of ϕ are distinct. Since any cubic apolar to ϕ is a linear combination of their cubes (§ 53),

$$f_i=c_{i1}\alpha_x^3+c_{i2}\beta_x^3+c_{i3}\gamma_x^3 \qquad (i=1, 2, 3).$$

Since the determinant $|c_{ij}|$ is not zero, suitable linear combinations of the f's give $\alpha_x^3, \beta_x^3, \gamma_x^3$. Hence by a linear transformation on ξ, η, ζ (i. e., by choice of a new triangle of reference), we may take *

$$\rho\xi=\alpha_x^3, \qquad \rho\eta=\beta_x^3, \qquad \rho\zeta=\gamma_x^3.$$

The line $\xi=0$ is an inflexion tangent, likewise $\eta=0$ and $\zeta=0$. In addition to the resulting three inflexion points, there are no others. For, at an inflexion point three consecutive points are collinear, so that (1) gives $\phi=\phi_x^3=0$. In the present

* We now have the formulas in the second part of § 54, where now α_x^3 is the actual, not a symbolic, expression of f_1, etc.

case there are therefore exactly three inflexion points and they are collinear.

56. Any Rational Plane Cubic Curve has a Double Point. Let P_x denote the point (ξ, η, ζ) determined by the pair of parameters x_1, x_2. If the ratios x_1/x_2 and y_1/y_2 are distinct and yet P_x coincides with P_y, then P_x is a double point. For, any straight line (2), § 55, through P_x meets the curve in only the three points whose pairs of parameters satisfy the cubic equation (3), and since two of these pairs give the same point P_x, the line meets the curve in a single further point. Hence there is a double point $P_x = P_y$ if and only if there are two distinct ratios x_1/x_2 and y_1/y_2 such that (1) holds identically in z_1, z_2.

Let Q be the quadratic form which vanishes for the pairs of parameters x_1, x_2 and y_1, y_2 giving a double point. By (1), and the first theorem in § 53, Q is apolar to $\phi_x{}^2\phi_z$ for z_1, z_2 arbitrary. Write $\phi'_x{}^3$ as a symbolic notation for ϕ, alternative to $\phi_x{}^3$. Applying the argument made in § 54 for three cubics to two quadratics, we see that the unique quadratic (apart from a constant factor) which is apolar to both $\phi_x{}^2\phi_z$ and $\phi'_x{}^2\phi'_w$ is their Jacobian

$$J = (\phi\phi')\phi_x\phi'_x \cdot \phi_z\phi'_w.$$

Since ϕ and ϕ' are equivalent symbols, their interchange must leave J unaltered. Hence

$$J = \tfrac{1}{2}(\phi\phi')\phi_x\phi'_x\{\phi_z\phi'_w - \phi'_z\phi_w\}.$$

The quantity in brackets equals $(\phi\phi')(zw)$ by (1), § 40. Discarding the constant factor $\frac{1}{2}(zw)$, we may take

$$Q = (\phi\phi')^2\phi_x\phi'_x$$

as the desired quadratic form. This is the Hessian of ϕ. Conversely, the pairs of values for which Q vanishes are the pairs of parameters of the unique double point of the curve.

57. Rational Space Quartic Curve. Such a curve is given by

$$\rho\xi = \alpha_x{}^4, \qquad \rho\eta = \beta_x{}^4, \qquad \rho\zeta = \gamma_x{}^4, \qquad \rho\omega = \delta_x{}^4,$$

where the four binary quartics are linearly independent. By § 54, there is a unique quartic ϕ apolar to each of the four. As in § 55, four points P_x, P_y, P_z, P_w on the curve are coplanar if and only if

$$\phi_x\phi_y\phi_z\phi_w=0.$$

Thus $\phi=0$ gives the four points at which the osculating plane meets the curve in four consecutive points. It may be shown that the values $x_1^{(i)}$, $x_2^{(i)}$ for which the Hessian of ϕ vanishes give the four points $P_{x^{(i)}}$ on the curve the tangents at which meet the curve again.

Fundamental Systems of Covariants of Binary Forms
§§ 58–63

58. Linear Forms. A linear form α_x is its own symbolic representation. If $\alpha_x=\beta_x$, then $(\alpha\beta)=0$. Hence the only covariants of α_x are products of its powers by constants. A fundamental system of covariants of n linear forms is evidently given by the forms and the $\frac{1}{2}n(n-1)$ invariants of type $(\alpha\beta)$, where α_x and β_x are two of the forms.

59. Quadratic Form. A covariant K of a single quadratic

$$f=\alpha_x^2=\beta_x^2=\ldots$$

may have no factor of type $(\alpha\beta)$ and then it is

$$\alpha_x^2\beta_x^2\gamma_x^2\ \ldots=f^k,$$

or may have the factor $(\alpha\beta)$ and hence the further factor $(\alpha\beta)$, $(\alpha\gamma)(\beta\delta)$, $(\alpha\gamma)\beta_x$, or $\alpha_x\beta_x$, including the possibility $\delta=\gamma$. In the first case, $K=(\alpha\beta)^2K_1$, where K_1 is a covariant to which the same argument may be applied. Now $(\alpha\gamma)=\alpha_y$ if $y_1=\gamma_2$, $y_2=-\gamma_1$. Hence in the last three cases, K has a factor of the type

$$\theta=(\alpha\beta)\alpha_y\beta_z,$$

where α_y is either α_x or a new mode of writing $(\alpha\gamma)$, and similarly β_z is either β_x or a new mode of writing $(\beta\delta)$.

Interchanging the equivalent symbols α and β, we get

$$\theta=(\beta\alpha)\beta_y\alpha_z=\tfrac{1}{2}(\alpha\beta)(\alpha_y\beta_z-\beta_y\alpha_z)=\tfrac{1}{2}(\alpha\beta)^2(yz),$$

by (1), § 40. We are thus led to the first case. Hence the fundamental system of covariants of f is composed of f and its discriminant.

EXERCISES

1. The fundamental system for $f=a_x^2=b_x^2$ and $l=\alpha_x=\beta_x$ is f, l, $(ab)^2$, $(a\alpha)^2$, $(a\alpha)a_x$.

2. The fundamental system for $f=a_x^2=b_x^2$ and $\phi=\alpha_x^2=\beta_x^2$ is f, ϕ, $(ab)^2$, $(\alpha\beta)^2$, $(a\alpha)^2$, $(a\alpha)a_x\alpha_x$. Hint:

$$(a\alpha)(a\beta)\alpha_z\beta_y=(a\alpha)^2\beta_y\beta_z-\tfrac{1}{2}(\alpha\beta)^2a_ya_z,$$

as proved by multiplying together the identities (Ex. 6, § 40)

$$(\alpha\beta)a_y\equiv(a\beta)\alpha_y-(a\alpha)\beta_y,\quad(\alpha\beta)a_z\equiv(a\beta)\alpha_z-(a\alpha)\beta_z,$$

and noting that α and β are equivalent symbols.

60. Theorems on Transvectants. In the expression (4), § 52, for a transvectant, each summand taken without the prefixed numerical factor is called a *term* of the transvectant. In the first transvectant (3), § 52, the difference of the two terms is

$$\{(\beta\gamma)\alpha_\xi-(\alpha\gamma)\beta_\xi\}\gamma_\xi^2=\{(\beta\alpha)\gamma_\xi\}\gamma_\xi^2,$$

by Ex. 6, § 40, and is the negative of the 0th transvectant (viz., product) of $(\alpha\beta)$ and γ_ξ^3. The act of removing a factor α_ξ and a factor β_ξ from a product and multiplying by the factor $(\alpha\beta)$ is called a *convolution* (*Faltung*). We have therefore an illustration of the following

LEMMA. *The difference between any two terms of a transvectant equals a sum of terms each a term of a lower transvectant of forms obtained by convolution* from the two given forms.*

Consider the rth transvectant of

$$f=P\alpha_\xi^{(1)}\ \ldots\ \alpha_\xi^{(k)},\qquad\phi=Q\beta_\xi^{(1)}\ \ldots\ \beta_\xi^{(l)},$$

where P and Q are products of determinantal factors. Then PQ is a factor of each term of the transvectant. Any two terms T and T' differ only as to the arrangements of the α's and the β's. Hence T' can be derived from T by a permuta-

* Including the case of no convolution, as γ_ξ^3 from itself, in the above example.

tion on the α's and one on the β's, and hence by successive interchanges of two α's and successive interchanges of two β's. Any such interchange is said to replace a term by an adjacent term. For example, the two terms of (3), § 52, are adjacent, each being derived from the other by the interchange of α with β. Between T and T' we may therefore insert terms $T_1, \ldots, T_n$ such that any term of the series $T, T_1, T_2, \ldots, T_n, T'$ is adjacent to the one on either side of it. Since

$$T-T'=(T-T_1)+(T_1-T_2)+\ldots+(T_{n-1}-T_n)+(T_n-T'),$$

it suffices to prove the lemma for adjacent terms.

The interchange of two α's or two β's affects just two factors of a term of (4), § 52. The types of adjacent terms are *

$$C(\alpha'\beta')(\alpha''\beta''), \qquad C(\alpha'\beta'')(\alpha''\beta');$$

$$C(\alpha'\beta')\beta''_\xi, \qquad C(\alpha'\beta'')\beta'_\xi;$$

where β' and β'' were interchanged. The difference of the last two terms is seen to equal $C(\beta''\beta')\alpha'_\xi$ by the usual identity. The latter is evidently a term of the $(r-1)$th transvectant of f and $(\beta''\beta')\phi/\{\beta''_\xi\beta'_\xi\}$, which is obtained from ϕ by one convolution.

The difference of the first two adjacent terms equals $C(\alpha'\alpha'')(\beta'\beta'')$, since

$$(\alpha'\alpha'')(\beta'\beta'')-(\alpha'\beta')(\alpha''\beta'')+(\alpha'\beta'')(\alpha''\beta')\equiv\frac{1}{2}\begin{vmatrix} \alpha'_1 & \alpha''_1 & \beta'_1 & \beta''_1 \\ \alpha'_2 & \alpha''_2 & \beta'_2 & \beta''_2 \\ \alpha'_1 & \alpha''_1 & \beta'_1 & \beta''_1 \\ \alpha'_2 & \alpha''_2 & \beta'_2 & \beta''_2 \end{vmatrix}=0,$$

as shown by Laplace's development. The same relation follows also from the identity just used by taking $\xi_1=-\alpha''_2$, $\xi_2=\alpha''_1$. The resulting difference is a term of the $(r-2)$th transvectant of

$$(\alpha'\alpha'')\frac{f}{\alpha'_\xi\alpha''_\xi}, \qquad (\beta'\beta'')\frac{\phi}{\beta'_\xi\beta''_\xi},$$

which are derived from f and ϕ by a convolution.

* A pair $C(\alpha'\beta')\alpha''_\xi$, $C(\alpha''\beta')\alpha'_\xi$, obtained by interchanging α' and α'', is essentially of the second type.

The Lemma leads to a more important result. By the proof leading to (4), § 52, the coefficient of each term of a transvectant is $1/N$, if N is the number of terms. Just as $S=\frac{1}{2}(T_1+T_2)$ implies $S-T_1=\frac{1}{2}(T_2-T_1)$, so

$$S=\frac{1}{N}(T_1+\ldots+T_N)$$

implies

$$S-T_1=\frac{1}{N}\{(T_2-T_1)+\ldots+(T_n-T_1)\}.$$

Hence *the difference between a transvectant and any one of its terms equal a sum of terms each a term of a lower transvectant of forms obtained by convolution from the two given forms.*

Each term of a lower transvectant may be expressed, by the same theorem, as the sum of that transvectant and terms of still lower transvectants, etc. Finally, when we reach a 0th transvectant, i.e., the product of the two forms, the only term is that product. Hence we have the fundamental

THEOREM. *The difference between any transvectant and any one of its terms is a linear function of lower transvectants of forms obtained by convolution from the two given forms.*

For example, from (3), § 52, and the result preceding the Lemma, we have

$$(\beta\gamma)\alpha_\xi\gamma_\xi^2=(\alpha_\xi\beta_\xi,\ \gamma_\xi^3)^1-\tfrac{1}{2}((\alpha\beta),\ \gamma_\xi^3)^0,$$

and $(\alpha\beta)$ is derived from $\alpha_\xi\beta_\xi$ by one convolution.

61. Irreducible Covariants of Degree m Found by Induction. Let

$$f=\alpha_x^n=\beta_x^n=\ldots=\lambda_x^n$$

be the binary n-ic whose fundamental system of covariants is desired. Since a term with the factor $(\alpha\beta)$ is of degree at least two in the coefficients of f, the only covariants of degree unity are kf, where k is a numerical constant. We shall say that f is the only irreducible covariant of degree unity, and that f, $K_1, \ldots, K_s$ form a complete set of irreducible covariants of degrees $<m$ if every covariant of degree $<m$ is a poly-

nomial in $f, \ldots, K_s$ with numerical coefficients. Given the latter, we seek the irreducible covariants of degree m.

A covariant of degree m is a polynomial in the $(\alpha\beta)$ and the α_x such that each term contains m letters $\alpha, \beta, \gamma, \ldots$. Let T_m be one of the terms with its numerical factor suppressed. Let $\alpha, \beta, \ldots, \kappa, \lambda$ be the m letters occurring in T_m, so that

$$T_m = P(\alpha\lambda)^a(\beta\lambda)^b \ldots (\kappa\lambda)^k\lambda_x^l \qquad (a+b+\ldots+k+l=n),$$

where P involves only $\alpha, \beta, \ldots, \kappa$. Then

$$T_{m-1} = P\alpha_x^a\beta_x^b \ldots \kappa_x^k$$

is a covariant of degree $m-1$. Evidently T_m is a term of

$$(T_{m-1}, \quad \lambda_x^n)^r \qquad (r=n-l),$$

since it is obtained by $r=a+b+\ldots+k$ convolutions from $T_{m-1}\lambda_x^n$. By the final theorem in § 60,

$$T_m = (T_{m-1}, f)^r + \sum_{j=0}^{r-1} c_j(\overline{T}_{m-1}, f)^j,$$

where the c_j are numerical constants, and each $\overline{T}_{m-1}$ is derived from T_{m-1} by convolutions and hence is a covariant of degree $m-1$. But the covariant of degree m was a linear function of the various T_m. Hence *every covariant of degree m of f is a linear function of transvectants $(C_{m-1}, f)^k$ of covariants C_{m-1} of order $m-1$ with f.* Such a transvectant is zero if $k>n$, in view of the order of f. Moreover, it suffices by (5), § 52, to employ the C_{m-1} which are products of powers of $f, K_1, \ldots, K_s$. Hence the covariants of degree m are linear functions of a finite number of transvectants.

In the examination of these transvectants $(C_{m-1}, f)^k$, we first consider those with $k=1$, then those with $k=2$, etc. *We may discard any $(C_{m-1}, f)^k$ for which C_{m-1} has a factor ϕ, of order $\geqq k$, which is a product of powers of $f, K_1, \ldots, K_s$, and of degree $<m-1$.* For, if T is a term of $(\phi, f)^k$, and if $C_{m-1}=q\phi$, then T is obtained by k convolutions of ϕf, and qT by the same k convolutions of $q\phi f$, not affecting q. Hence qT is a term of $(q\phi, f)^k$. Hence

$$(C_{m-1}, f)^k = qT + \sum_{j=0}^{k-1} c_j(\overline{C}_{m-1}, f)^j.$$

But the terms of the last sum have by hypothesis been considered previously, while the covariants q and T are of degree * $<m$ and hence are expressible in terms of $f, K_1, \ldots, K_s$.

62. Binary Cubic Form. The only irreducible covariant of degree one of

$$f=\alpha_x^3=\beta_x^3=\gamma_x^3$$

was shown to be f. The only covariants of degree two are

$$(\alpha\beta)^r\alpha_x^{3-r}\beta_x^{3-r} \qquad (r=0,\ 1,\ 2,\ 3).$$

This vanishes identically if r is odd. If $r=0$, we have f^2, which is reducible. Hence the only irreducible covariant of degree two is

$$(\alpha\beta)^2\alpha_x\beta_x=(f,f)^2=\text{Hessian } H \text{ of } f.$$

To find the irreducible covariants of degree $m=3$, we have $C_{m-1}=H$ or f^2. In the second case, C_{m-1} has the factor f of degree $<m-1$ and order $3\geqq k$ (since we cannot remove by convolution more than three factors from the second function f in the transvectant). Hence we may discard $C_{m-1}=f^2$. It remains to consider $(H,f)^k$, $k=1, 2$. Now

$$(H,f)=(\alpha\beta)^2(\alpha\gamma)\beta_x\gamma_x^2=\text{Jacobian } J \text{ of } H \text{ and } f$$

is irreducible, being of order and degree three and hence not a polynomial in f and H. Next,

$$(H,f)^2=(\alpha\beta)^2(\alpha\gamma)(\beta\gamma)\gamma_x=P(\alpha\beta)\gamma_x, \qquad P=(\alpha\beta)(\alpha\gamma)(\beta\gamma).$$

Interchanging α with γ, we get $P(\beta\gamma)\alpha_x$. Interchanging β with γ, we get $P(\gamma\alpha)\beta_x$. Hence

$$(H,f)^2=\tfrac{1}{3}P\{(\alpha\beta)\gamma_x+(\beta\gamma)\alpha_x+(\gamma\alpha)\beta_x\}=0.$$

The irreducible covariants of degree three or less are therefore f, H, J.

To find those of degree $m=4$, we have $C_{m-1}=f^3, fH, J$,

* This is evident for the factor q of C_{m-1}. Since ϕ is of degree $<m-1$, the term T of $(\phi, f)^k$ involves fewer than $m-1+1$ symbols $\alpha, \beta, \ldots$, and hence is of degree $< m$.

of which the first two may be discarded as before. It remains to consider $(J, f)^k$, for $k=1, 2, 3$. By § 52,

$$(J, f)=(\alpha\beta)^2(\alpha\gamma)(\beta_x\gamma_x^2,\ \delta_x^3)$$
$$=(\alpha\beta)^2(\alpha\gamma)\{\tfrac{1}{3}(\beta\delta)\gamma_x^2\delta_x^2+\tfrac{2}{3}(\gamma\delta)\beta_x\gamma_x\delta_x^2\}.$$

Replacing $(\beta\delta)\gamma_x$ by $(\gamma\delta)\beta_x+(\beta\gamma)\delta_x$, and noting that

$$(\alpha\beta)^2(\alpha\gamma)(\beta\gamma)\gamma_x\delta_x^3=(H, f)^2\cdot f=0,$$

we get

$$(J, f)=(\alpha\beta)^2(\alpha\gamma)(\gamma\delta)\beta_x\gamma_x\delta_x^2.$$

Interchange γ and δ. Hence

$$(J, f)=\tfrac{1}{2}(\alpha\beta)^2(\gamma\delta)\beta_x\gamma_x\delta_x\{(\alpha\gamma)\delta_x+(\delta\alpha)\gamma_x\}.$$

The quantity in brackets equals $-(\gamma\delta)\alpha_x$. Hence

$$(J, f)=-\tfrac{1}{2}(\alpha\beta)^2(\gamma\delta)^2\alpha_x\beta_x\gamma_x\delta_x=-\tfrac{1}{2}H^2.$$

Denoting H by $h_x^2=h'^2_x$, we have

$$J=(h_x^2,\ \alpha_x^3)=(h\alpha)h_x\alpha_x^2, \qquad f=\beta_x^3,$$
$$(J, f)^2=(h\alpha)(h\beta)(\alpha\beta)\alpha_x\beta_x+c((h\alpha)^2\alpha_x, f),$$

by the theorem in § 60. Here $\bar{J}=(h\alpha)^2\alpha_x=(H, f)^2=0$. Since the first term is changed in sign when α and β are interchanged, we have $(J, f)^2=0$.

For the third case,

$$(J, f)^3=((\alpha\beta)^2(\alpha\gamma)\beta_x\gamma_x^2,\ \delta_x^3)^3=(\alpha\beta)^2(\alpha\gamma)(\beta\delta)(\gamma\delta)^2=D,$$

an invariant, evidently equal to $(H, H)^2$, the discriminant of H. Thus D is the discriminant of f (§§ 8, 30) and is not identically zero. Hence D is the only irreducible covariant of degree four.

We can now prove by induction that f, H, J and D form a complete set of irreducible covariants of degree $\leqq m \geqq 5$. Let this be true for covariants C_{m-1} of degree $\leqq m-1$. We may discard $(C_{m-1}, f)^k$ if C_{m-1} has the factor f or J, each of which is of order $3 \geqq k$ and of degree (1 or 3) less than $m-1$; and evidently also if it has the factor D. Hence $C_{m-1}=H^e$, $e \geqq 2$. If $k \leqq 2$, it has the factor H of order $2 \geqq k$ and degree $2<m-1$. It remains to consider $(H^e, f)^3$. If $e>2$, H^e has the factor

H^2 of order $4 \geqq 3$ and degree $4 < m-1$, since H^e is of degree $\geqq 6$. Finally,

$$(H^2, f)^3 = (h_x{}^2 h'^2{}_x,\ \alpha_x{}^3)^3 = (h\alpha)^2 (h'\alpha) h'_x = (h'^2{}_x,\ (h\alpha)^2 \alpha_x) = 0.$$

Hence f, H, J, D form a fundamental system of covariants (*cf.* § 30).

63. Higher Binary Forms. The concepts introduced by Gordan in his proof of the finiteness of the fundamental system of covariants of the binary p-ic enabled him to find * the system of 23 forms for the quintic, the system of 26 forms for the sextic, as well as to obtain in a few lines the system for the cubic (§ 62) and the quartic (§ 31). Fundamental systems for the binary forms of orders 7 and 8 have been determined by von Gall.†

Gordan's method yields a set of covariants in terms of which all of the covariants are expressible rationally and integrally, but does not show that a smaller set would not serve similarly. The method is supplemented by Cayley's theory ‡ of generating functions, which gives a lower limit to the number of covariants in a fundamental system.

64. Hermite's Law of Reciprocity. This law (§ 27) can be made self-evident by use of the symbolic notation. Let the form

$$\phi = \alpha_x{}^p = \beta_x{}^p = \ldots = a_0(x_1 - \rho_1 x_2)(x_1 - \rho_2 x_2) \ldots (x_1 - \rho_p x_2)$$

have a covariant of degree d,

$$K = a_0{}^d \Sigma (\rho_1 - \rho_2)^i (\rho_1 - \rho_3)^j (\rho_2 - \rho_3)^k \ldots (x_1 - \rho_1 x_2)^{l_1} \ldots (x_1 - \rho_p x_2)^{l_p},$$

so that each of the roots $\rho_1, \ldots, \rho_p$ occurs exactly d times in each product. Consider the binary d-ic

$$f = a_x{}^d = b_x{}^d = \ldots = c_0(x_1 - r_1 x_2) \ldots (x_1 - r_d x_2).$$

* Gordan, *Invariantentheorie*, vol. 2 (1887), p. 236, p. 275. *Cf.* Grace and Young, *Algebra of Invariants*, 1903, p. 122, p. 128, p. 150.

† *Mathematische Annalen*, vol. 17 (1880), vol. 31 (1888).

‡ For an introduction to it, see Elliott, *Algebra of Quantics*, 1895, p. 165, p. 247.

To the various powers, whose product is any one term of K,

$$(\rho_1-\rho_2)^i, \quad (\rho_1-\rho_3)^j, \quad (\rho_2-\rho_3)^k, \quad \ldots, \quad (x_1-\rho_1 x_2)^{l_1}, \quad (x_1-\rho_2 x_2)^{l_2}, \ldots,$$

we make correspond the symbolic factors

$$(ab)^i, \quad (ac)^j, \quad (bc)^k, \quad \ldots, \quad a_x^{l_1}, \quad b_x^{l_2}, \quad \ldots$$

of the corresponding covariant of f:

$$C=(ab)^i(ac)^j(bc)^k \ldots a_x^{l_1}b_x^{l_2}c_x^{l_3} \ldots,$$

of degree p (since there are p symbols a, b, c, . . . , corresponding to ρ_1, . . . , ρ_p) and having the same order $l_1+l_2+l_3+\ldots$ as K. Conversely, C determines K.

EXAMPLES

Let $p=2$. To $K=a_0^{2s}(\rho_1-\rho_2)^{2s}$ corresponds the invariant $C=(ab)^{2s}$ of degree 2 of $f=a_x^{2s}=b_x^{2s}$. Again, to the covariant $K\phi^t$ of ϕ corresponds the covariant $(ab)^{2s}\, a_x^t b_x^t$ of the form $a_x^{2s+t}=b_x^{2s+t}$.

Concomitants of Ternary Forms in Symbolic Notation, §§ 65–67

65. Ternary Form in Symbolic Notation. The general ternary form is

$$f=\Sigma\frac{n!}{r!s!t!}\, a_{rst}x_1^r x_2^s x_3^t,$$

where the summation extends over all sets of integers r, s, t, each $\geqq 0$, for which $r+s+t=n$.

We represent f symbolically by

$$f=\alpha_x^n=\beta_x^n \ldots, \qquad \alpha_x=\alpha_1 x_1+\alpha_2 x_2+\alpha_3 x_3, \ldots .$$

Only polynomials in α_1, α_2, α_3 of total degree n have an interpretation and

$$\alpha_1^r\alpha_2^s\alpha_3^t=a_{rst}.$$

Just as $\alpha_1\beta_2-\alpha_2\beta_1$ was denoted by $(\alpha\beta)$ in § 39, we now write

$$(\alpha\beta\gamma)=\begin{vmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{vmatrix}.$$

Under any ternary linear transformation

$$T: \qquad x_i = \xi_i X_1 + \eta_i X_2 + \zeta_i X_3 \qquad (i=1, 2, 3)$$

α_x becomes $\alpha_\xi X_1 + \alpha_\eta X_2 + \alpha_\zeta X_3$, and f becomes

$$\Sigma \frac{n!}{r!s!t!} A_{rst} X^r{}_1 X^s{}_2 X^t{}_3 = (\alpha_\xi X_1 + \alpha_\eta X_2 + \alpha_\zeta X_3) .$$

Thus α_x behaves like a covariant of index zero of f. Also

$$A_{rst} = \alpha_\xi{}^r \alpha_\eta{}^s \alpha_\zeta{}^t,$$

$$\begin{vmatrix} \alpha_\xi & \alpha_\eta & \alpha_\zeta \\ \beta_\xi & \beta_\eta & \beta_\zeta \\ \gamma_\xi & \gamma_\eta & \gamma_\zeta \end{vmatrix} = (\alpha\beta\gamma)(\xi\eta\zeta),$$

so that $(\alpha\beta\gamma)$ behaves like an invariant of index unity of f.

EXERCISES

1. The discriminant of a ternary quadratic form $\alpha_x{}^2$ is $\frac{1}{6}(\alpha\beta\gamma)^2$.

2. The Jacobian of $\alpha_x{}^l$, $\beta_x{}^m$, $\gamma_x{}^n$ is $lmn\,(\alpha\beta\gamma)\alpha_x{}^{l-1}\beta_x{}^{m-1}\gamma_x{}^{n-1}$.

3. The Hessian of $\alpha_x{}^n$ is the product of $(\alpha\beta\gamma)^2\alpha_x{}^{n-2}\beta_x{}^{n-2}\gamma_x{}^{n-2}$ by a constant.

4. A ternary cubic form $\alpha_x{}^3 = \beta_x{}^3 = \ldots$ has the invariants

$$(\alpha\beta\gamma)(\alpha\beta\delta)(\alpha\gamma\delta)(\beta\gamma\delta), \quad (\alpha\beta\gamma)(\alpha\beta\delta)(\alpha\gamma\epsilon)(\beta\gamma\phi)(\delta\epsilon\phi)^2.$$

66. Concomitants of Ternary Forms. If u_1, u_2, u_3 are constants,

$$u_x = u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$$

represents a straight line in the point-coördinates x_1, x_2, x_3. Since u_1, u_2, u_3 determine this line, they are called its line-coördinates. If we give fixed values to x_1, x_2, x_3 and let the line-coördinates u_1, u_2, u_3 take all sets of values for which $u_x = 0$, we obtain an infinite set of straight lines through the point (x_1, x_2, x_3). Thus, for fixed x's, $u_x = 0$ is the equation of the point (x_1, x_2, x_3) in line-coördinates.

Under the linear transformation T, of § 65, whose determinant $(\xi\eta\zeta)$ is not zero, the line $u_x = 0$ is replaced by

$$U_X = U_1 X_1 + U_2 X_2 + U_3 X_3 = 0,$$

in which

$$U_1 = \sum_{i=1}^{3} \xi_i u_i, \qquad U_2 = \sum_{i=1}^{3} \eta_i u_i, \qquad U_3 = \sum_{i=1}^{3} \zeta_i u_i.$$

The equations obtained by solving these define a linear transformation T_1 which expresses u_1, u_2, u_3 as linear functions of U_1, U_2, U_3 and which is uniquely determined * by the transformation T. Two sets of variables x_1, x_2, x_3 and u_1, u_2, u_3, transformed in this manner, are called *contragredient*.

A polynomial $P(c, x, u)$ in the two sets of contragredient variables and the coefficients c of certain forms $f_i(x_1, x_2, x_3)$ is called a *mixed concomitant* of index λ of the f's if, for every linear transformation T of determinant $\Delta \neq 0$ on x_1, x_2, x_3 and the above defined transformation T_1 on u_1, u_2, u_3, the product of $P(c, x, u)$ by Δ^λ equals the same polynomial $P(C, X, U)$ in the new variables and coefficients C of the forms derived from the f's by the first transformation. For example, u_x is a concomitant of index zero of any set of forms.

In particular, if P does not involve the u's, it is a covariant (or invariant) of the f's. If it involves the u's, but not the x's, it is called a *contravariant* of the f's.

Since $U_1 = u_\xi$, $U_2 = u_\eta$, $U_3 = u_\zeta$, we see by the last formula in § 65, with γ replaced by u, that $(\alpha\beta u)$ behaves like a contravariant of index unity of α_x^n, and also like one of α_x^n, β_x^m.

For the linear forms α_x and β_x, $(\alpha\beta u)$ has an actual interpretation. For $f = \alpha_x^2 = \beta_x^2$, where

$$f = a_{200}x_1^2 + a_{020}x_2^2 + a_{002}x_3^2 + 2a_{110}x_1x_2 + 2a_{101}x_1x_3 + 2a_{011}x_2x_3,$$

it may be shown that

$$\begin{vmatrix} a_{200} & a_{110} & a_{101} & u_1 \\ a_{110} & a_{020} & a_{011} & u_2 \\ a_{101} & a_{011} & a_{002} & u_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix} = (\alpha\beta u)^2.$$

By equating to zero this determinant (the bordered discriminant of f), we obtain the line equation of the conic $f = 0$.

67. Theorem. *Every concomitant of a system of ternary forms is a polynomial in u_x and expressions of the types α_x, $(\alpha\beta\gamma)$, $(\alpha\beta u)$.*

* We have only to interchange the rows and columns in the matrix of T and then take the inverse of the new matrix to obtain the matrix of the transformation T_1. Similarly, x_1, x_2 are contragredient with u_1, u_2, if we have T, § 40, and $u_1 = (\eta_2 U_1 - \xi_2 U_2)/(\xi\eta)$, $u_2 = (-\eta_1 U_1 + \xi_1 U_2)/(\xi\eta)$.

A concomitant of the forms $f_i(x_1, x_2, x_3)$ is evidently a covariant of the enlarged system of forms f_i and u_x. We may therefore restrict attention to covariants. In the proof of the corresponding theorem for binary forms, we used the operator (1), § 42. Here we employ an operator V composed of six terms each a partial differentiation of the third order:

$$V=\begin{vmatrix} \frac{\partial}{\partial\xi_1} & \frac{\partial}{\partial\xi_2} & \frac{\partial}{\partial\xi_3} \\ \frac{\partial}{\partial\eta_1} & \frac{\partial}{\partial\eta_2} & \frac{\partial}{\partial\eta_3} \\ \frac{\partial}{\partial\zeta_1} & \frac{\partial}{\partial\zeta_2} & \frac{\partial}{\partial\zeta_3} \end{vmatrix}=\frac{\partial^3}{\partial\xi_1\partial\eta_2\partial\zeta_3}-\cdots,$$

the determinant being symbolic. It may be shown as in § 43 that

$$V(\xi\eta\zeta)^n=n(n+1)(n+2)(\xi\eta\zeta)^{n-1}.$$

As in § 44, the result of applying V^r to a product of k factors of the type α_ξ, l factors of the type β_η, and m factors of the type γ_ζ, is a sum of terms each containing $k-r$ factors α_ξ, $l-r$ factors β_η, $m-r$ factors γ_ζ, and r factors of the type $(\alpha\beta\gamma)$.

For the case of an invariant I, the theorem can be proved without a device. In the notations of § 65, we have

$$I(A)=(\xi\eta\zeta)^\lambda I(a).$$

Each A is a product of factors α_ξ, α_η, α_ζ. Hence $I(A)$ equals a sum of terms each with λ factors of the type α_ξ, λ of type α_η, and λ of type α_ζ. Operate on each member of the equation with V^λ. The left member becomes a sum of terms each a product of a constant and factors of type $(\alpha\beta\gamma)$. The right member becomes the product of $I(a)$ by a number not zero. Hence I equals a polynomial in the $(\alpha\beta\gamma)$.

For a covariant K, we have, by definition,

$$K(A, X)=(\xi\eta\zeta)^\lambda K(a, x).$$

Solving the equations of our transformation T in § 65, we get

$$(\xi\eta\zeta)X_1=x_1(\eta_2\zeta_3-\eta_3\zeta_2)+x_2(\eta_3\zeta_1-\eta_1\zeta_3)+x_3(\eta_1\zeta_2-\eta_2\zeta_1),$$

etc. Replacing x_1 by $y_2z_3-y_3z_2$, x_2 by $y_3z_1-y_1z_3$, and x_3 by $y_1z_2-y_2z_1$, we get

$$(\xi\eta\zeta)X_1=y_\eta z_\zeta-y_\zeta z_\eta,$$

$$(\xi\eta\zeta)X_2=y_\zeta z_\xi-y_\xi z_\zeta,$$

$$(\xi\eta\zeta)X_3=y_\xi z_\eta-y_\eta z_\xi.$$

Our relation for a covariant K of order ω now becomes

$$\Sigma(\text{product of factors } \alpha_\xi,\ y_\xi,\ z_\xi,\ \alpha_\eta,\ \ldots,\ z_\zeta)=(\xi\eta\zeta)^{\lambda+\omega}K(a,\ x),$$

each term on the left having $\lambda+\omega$ factors with the subscript ξ, etc. Apply the operator V to the left member. We obtain a sum of terms with one determinantal factor $(\alpha\beta\gamma)$, $(\alpha\beta y)$ or $(\alpha yz)\equiv\alpha_x$, and with $\lambda+\omega-1$ factors with the subscript ξ, etc. The result may be modified so that the undesired factor $(\alpha\beta y)$ shall not occur. For, it must have arisen by applying V to a term with a factor like $\alpha_\xi\beta_\eta y_\zeta$ and hence (by the formulas for the X_i) with a further factor z_η or z_ξ. Consider therefore the term $C\alpha_\xi\beta_\eta y_\zeta z_\eta$ in the initial result. Then the term $-C\alpha_\xi\beta_\eta y_\eta z_\zeta$ must occur. By operating on these with V, we get $C(\alpha\beta y)z_\eta$, $-C(\alpha\beta z)y_\eta$, respectively, whose sum equals

$$C\{(\beta yz)\alpha_\eta-(\alpha yz)\beta_\eta\}\equiv C(\beta_x\alpha_\eta-\alpha_x\beta_\eta),$$

as shown by expanding, according to the elements of the last row,

$$\begin{vmatrix} \alpha_1 & \beta_1 & y_1 & z_1 \\ \alpha_2 & \beta_2 & y_2 & z_2 \\ \alpha_3 & \beta_3 & y_3 & z_3 \\ \alpha_\eta & \beta_\eta & y_\eta & z_\eta \end{vmatrix}\equiv 0.$$

The modified result is therefore a sum of terms each with one factor of type $(\alpha\beta\gamma)$ or α_x and with $\lambda+\omega-1$ factors with subscript ξ, etc.

Applying V in succession $\lambda+\omega$ times and modifying the result at each step as before, we obtain as a new left member a sum of terms each with $\lambda+\omega$ factors of the types $(\alpha\beta\gamma)$ and α_x only. From the right member we obtain nK, where n is a number $\neq 0$. Hence the theorem is proved.

68. Quaternary Forms. For $\alpha_x = \alpha_1 x_1 + \ldots + \alpha_4 x_4$,

$$f = \alpha_x^n = \beta_x^n = \gamma_x^n = \delta_x^n$$

has the determinant $(\alpha\beta\gamma\delta)$ of order 4 as a symbolic invariant of index unity. Any invariant of f can be expressed as a polynomial in such determinantal factors; any covariant as a polynomial in them and factors of type α_x. In the equation $u_x = 0$ of a plane, $u_1, \ldots, u_4$ are called plane-coördinates. The mixed concomitants defined as in § 66 are expressible in terms of u_x and factors like α_x, $(\alpha\beta\gamma\delta)$, $(\alpha\beta\gamma u)$. For geometrical reasons, we extend that definition of mixed concomitants to polynomials $P(c, x, u, v)$, where $v_1, \ldots, v_4$ as well as $u_1, \ldots, u_4$ are contragredient to $x_1, \ldots, x_4$. There may now occur the additional type of factor

$$(\alpha\beta uv) = (\alpha_1\beta_2 - \alpha_2\beta_1)(u_3v_4 - u_4v_3) + \ldots + (\alpha_3\beta_4 - \alpha_4\beta_3)(u_1v_2 - u_2v_1).$$

These six combinations of the u's and v's are called the line-coördinates of the intersection of the planes $u_x = 0$, $v_x = 0$. For instance, $(\alpha\beta uv)^2 = 0$ is the condition that this line of intersection shall touch the quadric surface $\alpha_x^2 = 0$.

We have not considered concomitants involving also a third set of variables $w_1, \ldots, w_4$, contragredient with the x's. For, in

$$u_1x_1 + \ldots + u_4x_4 = 0, \qquad v_1x_1 + \ldots + v_4x_4 = 0,$$
$$w_1x_1 + \ldots + w_4x_4 = 0,$$

$x_1, \ldots, x_4$ are proportional to the three-rowed determinants of the matrix of coefficients, so that (αuvw) is essentially α_x.

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